Total time: 75 minutes.
Total points: 100.

All answers need to be justified!
In this exam, $\mathbb{P}_n$ denotes the vector space of polynomials with degree no more than n.

Problem 1 (10 points). Determine the dimension of

$$W = \text{Span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \right\}$$

Write these vectors as the columns of a matrix

$$\begin{pmatrix} -1 & 0 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & -1 \end{pmatrix}$$

Then W is the column space of this matrix. Do row reductions:

$$\begin{pmatrix} -1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{pmatrix}$$

$$\begin{pmatrix} \underline{-1} & 0 & 3 \\ 0 & \underline{1} & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

There are 2 pivots. Therefore $\dim W = 2$.

Problem 2 (15 points). Given that

$$\mathcal{B} = \{t^2 + 1,\ t - 1, t^2 - 2t + 5\}$$

is a basis for $\mathbb{P}_2$. Find the $\mathcal{B}$-coordinates of $f(t) = 3t^2 + 2t$.

Call the $\mathcal{B}$-coordinates of $f(t) = 3t^2 + 2t$ (c_1, c_2, c_3). Then we have

$$3t^2 + 2t = c_1(t^2 + 1) + c_2(t - 1) + c_3(t^2 - 2t + 5)$$

(equality as polynomials). Comparing the constant, t, t^2 coefficients, we get

$$c_1 - c_2 + 5c_3 = 0$$

$$0c_1 + c_2 - 2c_3 = 2$$

$$c_1 + 0c_2 + c_3 = 3$$

Therefore we need to solve the linear system with augmented matrix

$$\begin{pmatrix} 1 & -1 & 5 & 0 \\ 0 & 1 & -2 & 2 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

Do row reductions:

$$\begin{pmatrix} 1 & -1 & 5 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 5 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 5 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 5/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

Therefore $\mathcal{B}$-coordinates of $f(t) = 3t^2 + 2t$ are $7/2, 1, -1/2$.

Problem 3 (15 + 5 = 20 points). Let

$$A = \begin{pmatrix} -1 & -2 & 3 \\ 0 & -2 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

(1) Find the eigenvalues and corresponding eigenspaces of A.
(2) Is A diagonalizable? If yes, diagonalize it; if not, explain why.

(1) To find eigenvalues:

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & -2 & 3 \\ 0 & -2-\lambda & 0 \\ 1 & -2 & 1-\lambda \end{pmatrix} = (-1)^{2+2}(-2-\lambda)\det \begin{pmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{pmatrix}$$

$$= -(\lambda+2)((-1-\lambda)(1-\lambda)-3) = -(\lambda+2)(\lambda^2-4) = -(\lambda+2)^2(\lambda-2)$$

Therefore we get two eigenvalues $\lambda_1 = -2$, $\lambda_2 = 2$.
For $\lambda_1 = -2$, solve $(A - \lambda_1 I)\mathbf{x} = 0$:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 1 & -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \underline{1} & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 2x_2 - 3x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases}, \quad \mathbf{x} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the corresponding eigenspace is $\text{Span}\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \}$.

For $\lambda_2 = 2$, solve $(A - \lambda_2 I)\mathbf{x} = 0$:

$$\begin{pmatrix} -3 & -2 & 3 \\ 0 & -4 & 0 \\ 1 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ -3 & -2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = 0 \\ x_3 \text{ free} \end{cases}, \quad \mathbf{x} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore the corresponding eigenspace is Span$\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$.

(2) A is diagonalizable because the total dimension of eigenspace is $2 + 1 = 3$. It can be diagonalized as

$$A = PDP^{-1}, \quad P = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 4 (15 points). You are given that $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}) = \begin{pmatrix} x_3 \\ 0 \\ 4x_1 \end{pmatrix}$$

is a linear transformation. Find the eigenvalues of T.

Take the standard basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Then

$$T(\vec{e}_1) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \quad T(\vec{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad T(\vec{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

and the eigenvalues of T agree with the eigenvalues of the matrix $[T]_{\mathcal{B}}$. To find the latter,

$$\det(B - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 4 & 0 & -\lambda \end{pmatrix} = (-1)^{2+2}(-\lambda) \det \begin{pmatrix} -\lambda & 1 \\ 4 & -\lambda \end{pmatrix}$$

$$= -\lambda(\lambda^2 - 4) = -\lambda(\lambda + 2)(\lambda - 2)$$

Therefore the eigenvalues of T are $0, -2, 2$.

Problem 5 ($5 \times 4 = 20$ points). Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} 0 \\ -2 \\ -3 \\ 1 \end{pmatrix}$$

Calculate the following:

$$\|\mathbf{x}+\mathbf{y}\|, \quad \mathbf{x}\cdot(2\mathbf{y}), \quad (\mathbf{x}\cdot\mathbf{z})\mathbf{y}, \quad \text{the normalization of } \mathbf{z}$$

$$\|\mathbf{x}+\mathbf{y}\| = \| \begin{pmatrix} 3 \\ -4 \\ 3 \\ 3 \end{pmatrix} \| = \sqrt{9+16+9+9} = \sqrt{43}$$

$$\mathbf{x}\cdot(2\mathbf{y}) = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 6 \\ 2 \end{pmatrix} = 4+6+0+4 = 14$$

$$\mathbf{x}\cdot\mathbf{z} = 0+6+0+2 = 8, \quad (\mathbf{x}\cdot\mathbf{z})\mathbf{y} = \begin{pmatrix} 16 \\ -8 \\ 24 \\ 8 \end{pmatrix}$$

$$\|\mathbf{z}\| = \sqrt{0+4+9+1} = \sqrt{14}$$

Therefore the normalization of $\mathbf{z}$ is

$$\frac{1}{\sqrt{14}} \begin{pmatrix} 0 \\ -2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2/\sqrt{14} \\ -3/\sqrt{14} \\ 1/\sqrt{14} \end{pmatrix}$$

Problem 6 ($10+10=20$ points). Let W be a subspace of $\mathbb{R}^3$ with a basis

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

(1) Apply the Gram-Schmidt process to get an orthogonal basis for W.

(2) Use your result of (1) to find the orthogonal projection of $\mathbf{y} = \begin{pmatrix} 3 \\ 0 \\ -8 \end{pmatrix}$ onto W.

(1) Call the two original vectors $\mathbf{x}_1, \mathbf{x}_2$.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} - \frac{4+5+3}{4+1+1}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

Therefore an orthogonal basis for W is $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$.

(2)

$$\begin{aligned} \text{proj}_W \mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{6+0-8}{4+1+1}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{-6+0-8}{4+9+1}\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2/3 \\ 1/3 \\ -1/3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 10/3 \\ -4/3 \end{pmatrix} \end{aligned}$$