MATH 3300 — QUIZ 8 (20230328) SOLUTION

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Total time: 10 minutes.

Problem 1 (10 points). You are given that $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(f) = f(t-1)$ is a linear transformation. (for example, if $f(t) = 2t+1$ then $T(f) = 2(t-1)+1$.) Find the eigenvalues and corresponding eigenspaces of T.

Take the standard basis $\mathcal{B} = \{1, t, t^2\}$. Then

$$T(1) = 1, \quad T(t) = t-1, \quad T(t^2) = (t-1)^2 = t^2 - 2t + 1$$

Therefore the $\mathcal{B}$-matrix for T is

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(the columns are the $\mathcal{B}$-coordinates of $T(1), T(t), T(t^2)$).

A is an upper-triangular matrix, and thus the eigenvalues are the diagonal entries. The only eigenvalue is $\lambda = 1$.

For $\lambda = 1$, solve $(A - \lambda I)\mathbf{x} = 0$:

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \end{pmatrix}$$

Its solutions are

$$\begin{cases} x_1 \text{ free} \\ x_2 = 0 \\ x_3 = 0 \end{cases}, \quad \mathbf{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the corresponding eigenspace is $\text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$.

Therefore the only eigenvalue of T is $\lambda = 1$, with eigenspace $\text{Span}\{1\}$.