

Total time: 10 minutes.

Problem 1 (10 points). Let

$$f_1 = t + t^2 - t^3, \quad f_2 = -1 - t + 2t^3, \quad f_3 = (t + 1)^2$$

be polynomials in \mathbb{P}_3 . Determine the dimension of $\text{Span}\{f_1, f_2, f_3\}$.

Take the standard basis \mathcal{B} of \mathbb{P}_3 . The coordinates of f_1, f_2, f_3 are

$$[f_1]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad [f_2]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \quad [f_3]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

It suffices to find the dimension of the span of the above coordinate vectors in \mathbb{R}^4 .

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

Do row reductions:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \underline{1} & -1 & 2 \\ 0 & \underline{-1} & 1 \\ 0 & 0 & \underline{3} \\ 0 & 0 & 0 \end{pmatrix}$$

(underlined are pivots) There are 3 pivots. This means the desired dimension is 3.