Total time: 75 minutes.
Total points: 100.

All answers need to be justified!

Problem 1 ($5 \times 4 = 20$ points). Let

$$A = \begin{pmatrix} -1 & -1 & -6 \\ 2 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

Determine whether the following matrix addition/multiplications are well-defined. If yes, calculate them.

$$2B - B^T, \quad AB, \quad BA, \quad B^3$$

$$2B - B^T = \begin{pmatrix} 6 & 0 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$$

AB is undefined because A has 3 columns but B has 2 rows.

$$BA = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & -6 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} -3+0 & -3+0 & -18+0 \\ -1+4 & -1+0 & -6+8 \end{pmatrix} = \begin{pmatrix} -3 & -3 & -18 \\ 3 & -1 & 2 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 0 \\ 5 & 4 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 19 & 8 \end{pmatrix}$$

Problem 2 (15 points). Find the inverse of the following matrix:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 0 & -3 \\ 1 & -1 & -1 \end{pmatrix}$$

Start from

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & -3 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Do row reductions:

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 4 & -1 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 3 & -1 & 6 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{pmatrix}$$

Therefore

$$A^{-1} = \begin{pmatrix} 3 & -1 & 6 \\ 1 & 0 & 1 \\ 2 & -1 & 4 \end{pmatrix}$$

Problem 3 ($5 \times 3 = 15$ points). Let

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & -1 & -6 \\ 2 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

Calculate the following:

$$\det(A), \quad \det(BC^T B^T), \quad \det(B^8)$$

$$\det(A) = 3 \det \begin{pmatrix} -1 & -6 \\ 0 & 4 \end{pmatrix} - 2 \det \begin{pmatrix} -1 & -6 \\ 2 & 4 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} = 3 \times (-4) - 2 \times (-4 + 12) + 2 = -26$$

$$\det(BC^T B^T) = \det(B) \cdot \det(C) \cdot \det(B) = 2 \times 1 \times 2 = 4$$

$$\det(B^8) = \det(B)^8 = 2^8 = 256$$

Problem 4 (15 points). Determine whether the following vectors are linearly dependent:

$$\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -2 \\ 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 5 \\ -5 \\ -3 \end{pmatrix}$$

Put these vectors as the columns of a matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & -2 & 5 \\ 2 & 3 & -5 \\ 0 & 1 & -3 \end{pmatrix}$$

Do row reductions:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 7 \\ 0 & 3 & -9 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -2 & 7 \\ 0 & 0 & 3/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} \underline{1} & 0 & 2 \\ 0 & \underline{-2} & 7 \\ 0 & 0 & \underline{3/2} \\ 0 & 0 & 0 \end{pmatrix}$$

(underlined are pivots) Every column has a pivot. Therefore $A\mathbf{x} = \mathbf{0}$ has no nontrivial solutions, that is, the given vectors are linearly independent.

Problem 5 ($10 \times 2 = 20$ points).
(1) Prove that
$$\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 + x_2 \geq 0\right\}$$
is not a subspace of $\mathbb{R}^2$.
Denote this set as S. We have $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ because $1 + 1 = 2 \geq 0$ but $(-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin S$ because $-1 + (-1) = -2 < 0$. Therefore S is not closed under scalar multiplication, and thus not a subspace.

(2) Prove that
$$\{f(t) \in \mathbb{P}_2 : f(2) = 2f(-3)\}$$
is a subspace of $\mathbb{P}_2$. (Here $\mathbb{P}_2$ is the vector space of all polynomials of degree no more than 2, including the zero polynomial.)
Denote this set as S.
1. The zero polynomial $f(t) = 0$ is in S because $f(2) = f(-3) = 0$ and thus $f(2) = 2f(-3)$ is satisfied.
2. If $f, g \in S$ then $f(2) = 2f(-3)$, $g(2) = 2g(-3)$. Therefore
$$(f + g)(2) = f(2) + g(2) = 2f(-3) + 2g(-3) = 2(f + g)(-3)$$
Therefore $f + g \in S$.
3. If $f \in S$, c is a scalar, then $f(2) = 2f(-3)$. Therefore
$$(cf)(2) = cf(2) = c \cdot 2f(-3) = 2(cf)(-3)$$
Therefore $cf \in S$.

Problem 6 ($12 + 3 = 15$ points). Let

$$A = \begin{pmatrix} 0 & -2 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 5 \end{pmatrix}$$

(1) Find a basis for $\operatorname{Nul} A$.
(2) Find the dimension of $\operatorname{Nul} A$.
Do row operations on A:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 3 \\ 2 & 0 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & -2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{1} & 0 & 1/2 & 5/2 \\ 0 & \underline{1} & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(underlined are pivots) Therefore the solutions to $A\mathbf{x} = \mathbf{0}$ are

$$\begin{cases} x_1 = -\frac{1}{2}x_3 - \frac{5}{2}x_4 \\ x_2 = \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

In parametric form,

$$\mathbf{x} = x_3 \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Therefore a basis for $\operatorname{Nul} A$ is

$$\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

The dimension of $\operatorname{Nul} A$ is 2.