1.1. System of linear equations

Def A linear equation in the variables $x_1, x_2, \cdots, x_n$ is

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $\underbrace{a_1, \cdots, a_n}_{\text{coefficients}}$, b are real/complex numbers

Ex x_1, x_2, x_3

$$2x_1 - x_2 + 3x_3 = -4$$

Def A system of linear equations is a collection of linear equations involving the same variables $x_1, \cdots, x_n$

Ex x_1, x_2

$$\begin{cases} x_1 + 2x_2 = 4 & \text{--- } ① \\ 3x_1 - 4x_2 = 2 & \text{--- } ② \end{cases}$$

Def A solution to a system of linear equations involving $x_1, \cdots, x_n$ is a list of numbers $s_1, \cdots, s_n$ such that every eq. is satisfied if $x_1, \cdots, x_n$ replaced by $s_1, \cdots, s_n$

To solve it,

$② + (-3) \cdot ①$ $\quad -10x_2 = -10$

$\Rightarrow x_2 = 1$

Substitute into ①.

$$x_1 = 4 - 2x_2 = 4 - 2 \times 1 = 2$$

$\Rightarrow$ Solution: $\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$

Ex Solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \text{--- } ① \\ -2x_1 - x_2 + 0x_3 = -1 & \text{--- } ② \\ x_1 - x_2 + 2x_3 = 2 & \text{--- } ③ \end{cases}$$

$② + 2 ①$ $\quad 3x_2 + 2x_3 = 3 \text{ ---- } ②'$

$③ + (-1) \cdot ①$ $\quad -3x_2 + x_3 = 0 \text{ ---- } ③'$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \text{--- } ①' \\ 3x_2 + 2x_3 = 3 & \text{--- } ②' \\ -3x_2 + x_3 = 0 & \text{--- } ③' \end{cases}$$

$③' + 1 \cdot ②'$ $\quad 3x_3 = 3$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_2 + 2x_3 = 3 \\ 3x_3 = 3 \end{cases}$$

upper triangular

$x_3 = 1$

$3x_2 = 3 - 2x_3 = 3 - 2 \times 1 = 1$, $x_2 = \frac{1}{3}$

$x_1 = 2 - 2x_2 - x_3 = 2 - 2 \times \frac{1}{3} - 1 = \frac{1}{3}$

backward substitution

Procedure.
Gaussian elimination

$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{1}{3} \\ x_3 = 1 \end{cases}$$

To solve it,

② + (−3)·① $-10x_2 = -10$

$\Rightarrow x_2 = 1$

Substitute into ①.

$$x_1 = 4 - 2x_2 = 4 - 2\times 1 = 2$$

$\Rightarrow$ solution: $\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$

Ex Solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \text{— ①} \\ -2x_1 - x_2 + 0\cdot x_3 = -1 & \text{— ②} \\ x_1 - x_2 + 2x_3 = 2 & \text{— ③} \end{cases}$$

② + 2·① $3x_2 + 2x_3 = 3$ ···· ②′

③ + (−1)·① $-3x_2 + x_3 = 0$ ···· ③′

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \text{— ①′} \\ 3x_2 + 2x_3 = 3 & \text{— ②′} \\ -3x_2 + x_3 = 0 & \text{— ③′} \end{cases}$$

③′ + 1·②′ $3x_3 = 3$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_2 + 2x_3 = 3 \\ 3x_3 = 3 \end{cases}$$

upper triangular

$x_3 = 1$

$3x_2 = 3 - 2x_3 = 3 - 2\cdot 1 = 1$, $x_2 = \frac{1}{3}$

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backward substitution

Procedure.

Gaussian elimination

$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{1}{3} \\ x_3 = 1 \end{cases}$$

Matrix notation

For a system of linear eqs.

m eqs, n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

augmented matrix

m rows, $n+1$ columns

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Coefficient matrix

m rows, n columns

Ex Solve by augmented matrix

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 \\ 1 & -1 & 2 & 2 \end{array}\right) \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

② ⇝ ② + 2①

③ ⇝ ③ + (−1)①

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & -3 & 1 & 0 \end{array}\right) \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

③ ⇝ ③ + 1·②

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

③ ⇝ $\frac{1}{3}$③

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

To solve it,

② + (−3)·① $-10x_2 = -10$

$\Rightarrow x_2 = 1$

Substitute into ①.

$x_1 = 4 - 2x_2 = 4 - 2\times 1 = 2$

$\Rightarrow$ Solution: $\begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$

Ex Solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \cdots ① \\ -2x_1 - x_2 + 0x_3 = -1 & \cdots ② \\ x_1 - x_2 + 2x_3 = 2 & \cdots ③ \end{cases}$$

② + 2 ① $3x_2 + 2x_3 = 3 \cdots ②'$

③ + (−1)·① $-3x_2 + x_3 = 0 \cdots ③'$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 & \cdots ①' \\ 3x_2 + 2x_3 = 3 & \cdots ②' \\ -3x_2 + x_3 = 0 & \cdots ③' \end{cases}$$

③' + 1·②' $3x_3 = 3$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_2 + 2x_3 = 3 \\ 3x_3 = 3 \end{cases}$$

upper triangular

Procedure. Gaussian elimination

$x_3 = 1$

$3x_2 = 3 - 2x_3 = 3 - 2\times 1 = 1, \quad x_2 = \frac{1}{3}$

$x_1 = 2 - 2x_2 - x_3 = 2 - 2\times\frac{1}{3} - 1 = \frac{1}{3}$

backward substitution.

$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{1}{3} \\ x_3 = 1 \end{cases}$$

① ⟶ ① + (−1) ③

② ⟶ ② + (−2) ③

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

② ⟶ $\frac{1}{3}$ ②

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

① ⟶ ① + (−2) ②

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

Ex

$$\begin{cases} x_1 - 2x_2 = 1 \\ 2x_1 - 4x_2 = 3 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -4 & 3 \end{array}\right)$$

② → ② + (−2)·①

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

→ "0 = 1"

$\Rightarrow$ no solution

Elementary row operations

- replace row i by row i + c row j $i \neq j$ c any number
- replace row i by c row i, $c \neq 0$
- interchange two rows.