Total time: 75 minutes.
Total points: 100.

Problem 1 ($5 \times 5 = 25$ points). Calculate derivatives / higher order derivatives.

(1) $\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\sin(x^2)}{1-x^3}\Big) = \frac{\cos(x^2)\cdot 2x(1-x^3) - \sin(x^2)\cdot(-3x^2)}{(1-x^3)^2}$

(2) $(x^2e^{-x})''$

$(x^2e^{-x})' = 2xe^{-x} - x^2e^{-x} = (2x-x^2)e^{-x}, \quad (x^2e^{-x})'' = (2-2x)e^{-x} - (2x-x^2)e^{-x}$

(3) $\big(\sin^{-1}(\sqrt{x}) - (\tan^{-1}x)^{10}\big)' = \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} - 10(\tan^{-1}x)^9\cdot\frac{1}{1+x^2}$

(4) $\big((x^2+1)^{\cos x}\big)' = (e^{\cos x\ln(x^2+1)})' = e^{\cos x\ln(x^2+1)}(-\sin x\ln(x^2+1) + \cos x\cdot\frac{1}{x^2+1}\cdot 2x)$

(5) $(3e^{2x})^{(20)} = 3\cdot 2^{20}e^{2x}$

Problem 2 ($7+3=10$ points).
(1) Compute the linear approximation of $f(x)=x^{2/3}$ at $x=8$.
(2) Use your previous result to approximate $7.998^{2/3}$.

(1)

$$f'(x)=\frac{2}{3}x^{-1/3}, \quad f(8)=4, \quad f'(8)=\frac{1}{3}$$

Therefore the linear approximation is

$$L(x)=4+\frac{1}{3}(x-8)$$

(2)

$$7.998^{2/3}=f(7.998)\approx L(7.998)=4+\frac{1}{3}\cdot(-0.002)$$

Problem 3 ($20 = 10 + 10$ points). Consider the implicit function $y(x)$ given by

$$\sin x + xy = y^5 - 1$$

(1) Find the tangent line to the graph at $(0, 1)$.

Take implicit differentiation,

$$\cos x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 5y^4\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\cos x + y = (5y^4 - x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x + y}{5y^4 - x}$$

At $(0, 1)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 0 + 1}{5 - 0} = \frac{2}{5}$$

Therefore the tangent line is

$$y - 1 = \frac{2}{5}x$$

(2) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (Your final answer should be expressed in terms of x and y)

Take $\frac{\mathrm{d}}{\mathrm{d}x}$ on the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x + y}{5y^4 - x}$, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(-\sin x + \frac{\mathrm{d}y}{\mathrm{d}x})(5y^4 - x) - (\cos x + y)(20y^3\frac{\mathrm{d}y}{\mathrm{d}x} - 1)}{(5y^4 - x)^2}$$

$$= \frac{(-\sin x + \frac{\cos x + y}{5y^4 - x})(5y^4 - x) - (\cos x + y)(20y^3\frac{\cos x + y}{5y^4 - x} - 1)}{(5y^4 - x)^2}$$

Problem 4 (15 points). A rectangular-shaped window of width 8cm and height 5cm is shown on the screen of a computer. A person starts to adjust the window, so that it is getting more narrow at a rate of 2cm/s and getting taller at a rate of 3cm/s. Is the area of the window getting larger or smaller at this moment? (Your answer should be justified)

Let $a(t), b(t)$ be the width and height of the window, and $S(t)$ be the area of the window. Then

$$S(t) = a(t)b(t)$$

$$S'(t) = a'(t)b(t) + a(t)b'(t)$$

At the given time (denoted t_0),

$$a(t_0) = 8, \quad b(t_0) = 5, \quad a'(t_0) = -2, \quad b'(t_0) = 3$$

Therefore

$$S'(t_0) = -2 \cdot 5 + 8 \cdot 3 = 14 > 0$$

Therefore the area of the window is getting larger.

Problem 5 (15 points). Sketch the graph of the function $f(x) = x^3 + x^2 + x$.

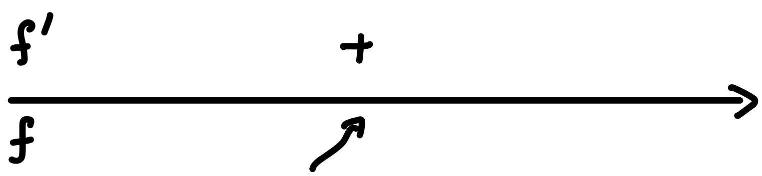
The domain of f is $(-\infty, \infty)$.

$$f'(x) = 3x^2 + 2x + 1$$

Setting $f'(x) = 0$ gives

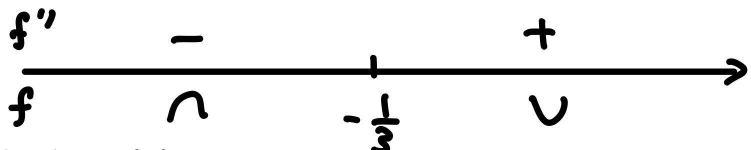
$$x = \frac{-2 \pm \sqrt{4 - 4 \cdot 3}}{-6}$$

where the quantity inside square root is negative. Therefore $f'(x)$ has no root.



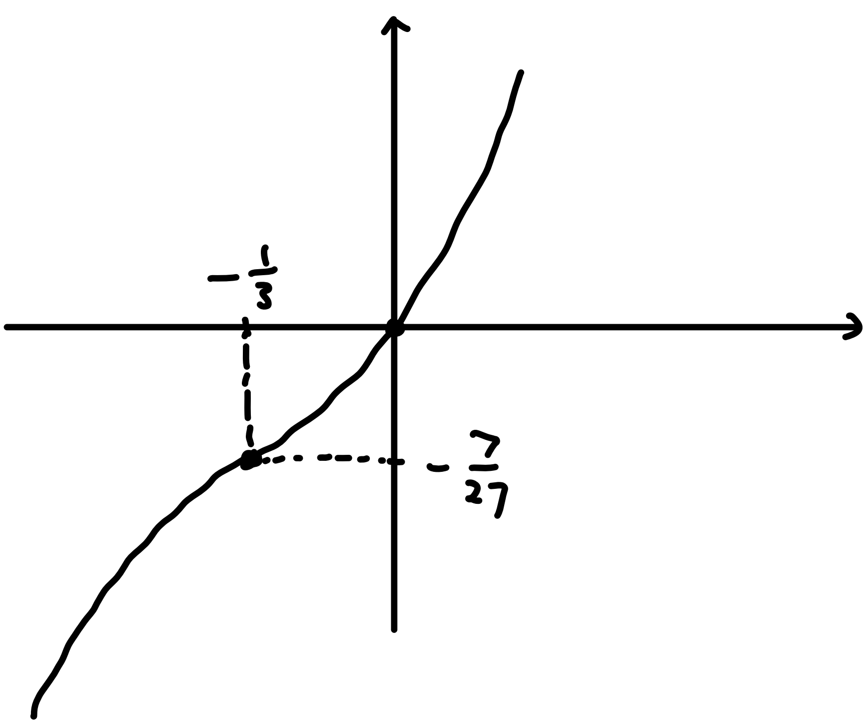
$$f''(x) = 6x + 2$$

Setting $f''(x) = 0$ gives $x = -\frac{1}{3}$.



Some special values of f:

$$f(-\frac{1}{3}) = -\frac{7}{27}, \quad f(0) = 0$$



Problem 6 (15 points). Find the global maximum and minimum of the function $f(x) = x + \frac{4}{x^2}$ on $[1, 3]$.

$$f'(x) = 1 - \frac{8}{x^3}$$

$f'(x) = 0$ gives $x = 2$, which is in $[1, 3]$.

$$f(2) = 3, \quad f(1) = 5, \quad f(3) = 3 + \frac{4}{9} = \frac{31}{9}$$

Therefore the global maximum is $f(1) = 5$, and the global minimum is $f(2) = 3$.