INSTRUCTOR: RUIWEN SHU

Total time: 75 minutes.
Total points: 100.

Problem 1 ($5 \times 4 = 20$ points). Determine limits.

(1) $$\lim_{x\to 1^-} \frac{x^2-4x+3}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x-3)}{x-1} = \lim_{x\to 1^-}(x-3) = -2$$

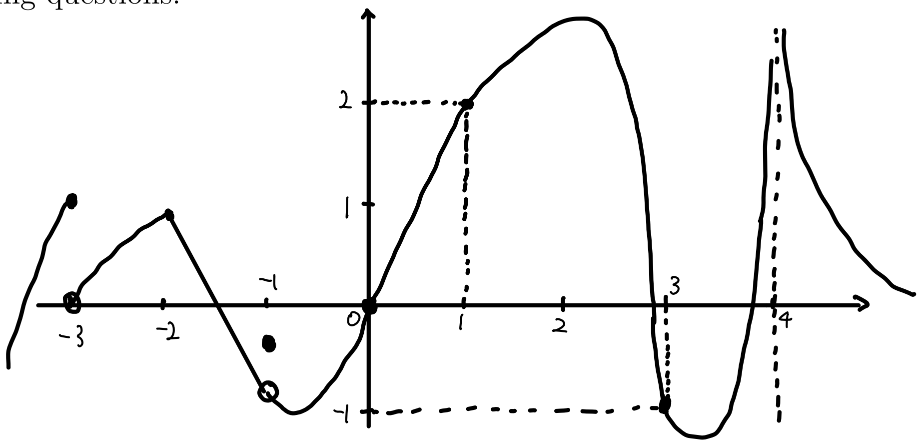
(2) $$\lim_{x\to 1^-} \frac{-2x-3}{|x-1|}$$

The numerator approaches -5, and the denominator approaches 0 while being positive. Therefore the answer is $-\infty$.

(3) $$\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{(x+2)-4}$$
$$= \lim_{x\to 2}(\sqrt{x+2}+2) = 4$$

(4) $$\lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{x^2+2x}\right) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{x(x+2)}\right) = \lim_{x\to 0}\frac{(x+2)-2}{x(x+2)} = \lim_{x\to 0}\frac{1}{x+2} = \frac{1}{2}$$

Problem 2 (5+5+5 = 15 points). The graph of the function $f(x)$ is given as below. Answer the following questions:



(1) Find the point(s) where $f(x)$ is discontinuous. (You only need to give an answer)

$$-3, -1, 4$$

(2) Find the point(s) where $f(x)$ is not differentiable. (You only need to give an answer)

$$-3, -2, -1, 4$$

(3) Approximate $f'(1)$ using the information in the picture in the best possible way.
We want an approximation

$$f'(1) \approx \frac{f(1+h) - f(1)}{h}$$

with (nonzero) $|h|$ as small as possible. The known point value closest to $x = 1$ is at $x = 0$. Therefore we take $h = -1$ and get

$$f'(1) \approx \frac{f(0) - f(1)}{-1} = \frac{0-2}{-1} = 2$$

Problem 3 (15 points). Find the derivative of the following function **by definition**.

$$f(x) = \frac{x}{x-1}$$

The domain of f is $(-\infty, 1) \cup (1, \infty)$. Take any x in the domain.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} = \lim_{h\to 0} \frac{(\frac{x+h}{x+h-1} - \frac{x}{x-1})(x+h-1)(x-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{(x+h)(x-1) - x(x+h-1)}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{x^2 + hx - x - h - x^2 - hx + x}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}$$

Problem 4 ($5 \times 4 = 20$ points). Calculate the following derivatives / higher order derivatives.

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\Big((x^3 - 2x)\sin x\Big) = (3x^2 - 2)\sin x + (x^3 - 2x)\cos x$$

(2) $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{t^4+1}\right) = \frac{2t(t^4+1) - t^2 \cdot 4t^3}{(t^4+1)^2} = \frac{-2t^5 + 2t}{(t^4+1)^2}$$

(simplification is not required)

(3) $$\left(x^4 - \frac{3}{x}\right)'' = \left(4x^3 + \frac{3}{x^2}\right)' = 12x^2 - \frac{6}{x^3}$$

(4) $$(\cos x)^{(54)} = ((\cos x)^{(52)})'' = (\cos x)'' = -\cos x$$

(since 52 is a multiple of 4)

Problem 5 (10 + 5 = 15 points). Given the function

$$f(x) = \begin{cases} 2x^2 + x & \text{when } x \leq -1 \\ -2x - 1 & \text{when } x > -1 \end{cases}$$

(1) Find $f'(x)$.
If $x < -1$, then $f(x) = 2x^2 + x$ and $f'(x) = 4x + 1$.
If $x > -1$, then $f(x) = -2x - 1$ and $f'(x) = -2$.
If $x = -1$, then (first calculate $f(-1) = 2(-1)^2 + (-1) = 1$)

$$\lim_{h\to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0^-} \frac{2(-1+h)^2 + (-1+h) - 1}{h} = \lim_{h\to 0^-} \frac{2 - 4h + h^2 - 1 + h - 1}{h} = -3$$

$$\lim_{h\to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0^-} \frac{-2(-1+h) - 1 - 1}{h} = \lim_{h\to 0^-} \frac{2 - 2h - 2}{h} = -2$$

These two one-sided limits are not equal. Therefore $f'(-1)$ DNE.
Therefore,

$$f'(x) = \begin{cases} 4x + 1 & \text{when } x < -1 \\ -2 & \text{when } x > -1 \end{cases}$$

(2) Find the tangent line to the graph of f at $(-2, f(-2))$.
Using part (1), $f'(-2) = 4(-2)+1 = -7$. $f(-2) = 2(-2)^2 + (-2) = 6$. Therefore the equation of the tangent line is

$$y - 6 = -7(x + 2)$$

Problem 6 ($5+5+5=15$ points). A car travels on a road. It starts from the origin at time $t=0$, and the distance between the car and the origin is given by

$$s(t)=t+2t^2-t^3, \quad t\geq 0$$

(1) At $t=0.5$, is the car moving forward or backward?

$$v(t)=1+4t-3t^2, \quad v(0.5)=1+4\cdot 0.5-3\cdot 0.5^2=2.25>0$$

Therefore at $t=0.5$ the car is moving forward.

(2) Find the acceleration $a(t)$ of the car.

$$a(t)=4-6t$$

(3) Find the time(s) at which the car is at rest.
We seek for values of t with $v(t)=0$.

$$-3t^2+4t+1=0, \quad t=\frac{-4\pm\sqrt{4^2-4\cdot(-3)\cdot 1}}{2(-3)}=\frac{-4\pm\sqrt{28}}{-6}$$

The root $\frac{-4+\sqrt{28}}{-6}$ is negative (by analysis or calculator) and should be discarded due to the requirement $t\geq 0$. Therefore the answer is

$$t=\frac{-4-\sqrt{28}}{-6}=\frac{2+\sqrt{7}}{3}$$