Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use MATLAB or the internet in this exam.

Problem 1. (30=15+15 points) Determine whether the following matrices are diagonalizable (using real numbers). Your answer should be justified.

(1)

$$A = \begin{pmatrix} 3 & 2 & -2 \\ -2 & -2 & 2 \\ -1 & -2 & 2 \end{pmatrix}$$

(2)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{pmatrix}$$

(1)

$$\begin{aligned}
\det(A - \lambda I) = &\det \begin{pmatrix} 3-\lambda & 2 & -2 \\ -2 & -2-\lambda & 2 \\ -1 & -2 & 2-\lambda \end{pmatrix} \\
=&(3-\lambda)(-2-\lambda)(2-\lambda) + 2\cdot 2\cdot(-1) + (-2)(-2)(-2) \\
&- (3-\lambda)\cdot 2\cdot(-2) - (-2-\lambda)\cdot(-1)\cdot(-2) - (2-\lambda)\cdot 2\cdot(-2) \\
=& -\lambda^3 + 3\lambda^2 + 4\lambda - 12 - 4 - 8 + 12 - 4\lambda + 2\lambda + 4 + 8 - 4\lambda \\
=& -\lambda^3 + 3\lambda^2 - 2\lambda \\
=& -\lambda(\lambda-1)(\lambda-2)
\end{aligned}$$

Therefore $0, 1, 2$ are eigenvalues of A. Since the 3×3 matrix A has 3 distinct eigenvalues, A is diagonalizable.

(2)

$$
\begin{aligned}
\det(A-\lambda I) = &\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & 2-\lambda & -1 \\ -2 & 2 & -1-\lambda \end{pmatrix} \\
=&(1-\lambda)\det\begin{pmatrix} 2-\lambda & -1 \\ 2 & -1-\lambda \end{pmatrix} \\
=&(1-\lambda)((2-\lambda)(-1-\lambda)+2) \\
=&(1-\lambda)(\lambda^2-\lambda) \\
=&-\lambda(\lambda-1)^2
\end{aligned}
$$

Therefore A has eigenvalues $\lambda_1 = 0$ with multiplicity 1, and $\lambda_2 = 1$ with multiplicity 2. For $\lambda_1 = 0$, its eigenspace can only have dimension 1. For $\lambda_2 = 1$, to get the eigenspace, we use row operations:

$$
\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ -2 & 2 & -2 & 0 \end{array}\right)
$$

$$
\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

There are two free variables, and therefore the eigenspace corresponding to $\lambda_2 = 1$ has dimension 2, coinciding with its multiplicity.
Therefore we conclude that A is diagonalizable.

Problem 2. (20=10+10 points) Let

$$
A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 5 & 0 & -3 \end{pmatrix}
$$

(1) Find all the eigenvalues of A, <u>including complex ones</u>.

(2) Find a complex eigenvector for <u>one of the non-real</u> eigenvalue.

(1)

$$\begin{aligned}\det(A-\lambda I) =& \det\begin{pmatrix} 3-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ 5 & 0 & -3-\lambda \end{pmatrix} \\ =&(-\lambda)\det\begin{pmatrix} 3-\lambda & -2 \\ 5 & -3-\lambda \end{pmatrix} \\ =&(-\lambda)((3-\lambda)(-3-\lambda)+10) \\ =&(-\lambda)(\lambda^2+1) \\ =&(-\lambda)(\lambda+i)(\lambda-i)\end{aligned}$$

Therefore, the complex eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -i$, $\lambda_3 = i$.

For the eigenvalue $\lambda_3 = i$, we use row operations:

$$\left(\begin{array}{ccc|c} 3-i & 0 & -2 & 0 \\ 0 & -i & 0 & 0 \\ 5 & 0 & -3-i & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3-i & 0 & -2 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore one complex eigenvector corresponding to $\lambda_3 = i$ is $\begin{pmatrix} 2 \\ 0 \\ 3-i \end{pmatrix}$.

Problem 3. (20=10+10 points) Let P_n be the vector space of polynomials of degree no more than n. Define the linear transformation T on P_2 by

$$T(p(t)) = p'(t)(t+1)$$

where $p'(t)$ is the derivative of $p(t)$ (you are given the fact that this is a linear transformation on P_2).

(1) Let $B = \{1, t, t^2\}$ be the standard basis of P_2. Compute $[T]_B$, the matrix for T relative to B.

(2) Show that 2 is an eigenvalue of T, and find a corresponding eigenvector.

(1)

$$\begin{aligned}T(1) &= 0\cdot(t+1) = 0 = 0\cdot 1 + 0\cdot t + 0\cdot t^2 \\ T(t) &= 1\cdot(t+1) = t+1 = 1\cdot 1 + 1\cdot t + 0\cdot t^2 \\ T(t^2) &= 2t\cdot(t+1) = 2t^2+2t = 0\cdot 1 + 2\cdot t + 2\cdot t^2\end{aligned}$$

Therefore

$$[T]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

(2) $[T]_B$ is an upper triangular matrix with diagonal elements $0, 1, 2$, and these are the eigenvalues of $[T]_B$. Therefore 2 is eigenvalue of $[T]_B$, and it is an eigenvalue of T. To get an eigenvector for $[T]_B$ corresponding to $\lambda = 2$, we solve $([T]_B - \lambda I)\mathbf{x} = 0$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore an eigenvector is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Therefore an eigenvector for T corresponding to $\lambda = 2$ is $1 \cdot 1 + 2 \cdot t + 1 \cdot t^2 = t^2 + 2t + 1$.

Problem 4. (15=10+5 points) Let

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ x \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ y \\ 3 \\ 1 \end{pmatrix} \right\}$$

be a set of vectors in $\mathbb{R}^4$, where x and y are unknown real numbers.

(1) Find the value of x and y such that S is an orthogonal set.

(2) With the choice of x and y in (1), what is the dimension of $\text{Span}(S)$? Justify your answer.

(1) Since every pair of distinct vectors in S is orthogonal, we get

$$2 + 0 - 2 + 0 = 0, \quad 2 + 0 - 6 + x = 0, \quad 4 + y + 3 + 0 = 0$$

which gives $x = 4$, $y = -7$.

(2) S is an orthogonal set, and therefore linearly independent. Since S has 3 elements, the dimension of $\text{Span}(S)$ is 3. (an alternative way is to check that these three vectors are linearly independent by using row operations.)

Problem 5. (15 points) Given that

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Construct an orthogonal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt process on S.

Denote the three vectors in S as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \frac{3+0+2}{1+0+4} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{pmatrix} -1 \\ 6 \\ 2 \end{pmatrix} - \frac{-1+0-4}{1+0+4} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} - \frac{-2+6+2}{4+1+1} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$$