MATH 310 Midterm Exam 3 Solution
29 Apr 2021, Instructor: Ruiwen Shu

Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use the internet in this exam.
(5) When proving a proposition, you may use conclusions which are significantly simpler than the current proposition.

Problem 1. (30=6+6+6+6+6 points) Determine whether the following functions are one-to-one and whether they are onto the codomain. You only need to give an answer, and no partial credit is given.

$$(1): \quad f:\mathbb{R}\to\mathbb{R}, \quad f(x)=x^2 \quad \text{not one-to-one, not onto}$$

$$(2): \quad f:\mathbb{R}\to\mathbb{R}, \quad f(x)=x^3 \quad \text{one-to-one, onto}$$

$$(3): \quad f:\mathbb{R}\to\mathbb{R}, \quad f(x)=x^3-x \quad \text{not one-to-one, onto}$$

$$(4): \quad f:\mathbb{Z}\to\mathbb{Z}, \quad f(x)=x^3 \quad \text{one-to-one, not onto}$$

$$(5): \quad f:\mathbb{R}\to\mathbb{Z}, \quad f(x)=\lceil x\rceil \quad \text{not one-to-one, onto}$$

Some explanation: (1)(2)(3) can be seen by sketching graphs. For a more rigorous justification,
(2): f is strictly increasing and thus one-to-one. $f(x)$ is continuous and goes to $+\infty$ as x approaches $+\infty$, goes to $-\infty$ as x approaches $-\infty$, and thus f is onto.
(3) $f(0)=f(1)=0$ and thus f is not one-to-one. For 'onto', similar to (2).
(4) f is one-to-one because it is a restriction of the function in (2). It is not onto because $2\notin \text{Rng}(f)$.
(5) f is not one-to-one because $f(0.1)=f(0.2)=1$. It is onto because $f(x)=x$ for any integer x.

Problem 2. (20=5+15 points) Let $f(x)=x^2-2x$ be a real function, and you are given that the range of f is $[-1,\infty)$.

(1) Prove that f is not one-to-one.

(2) Find some subset $D\subseteq\mathbb{R}$, such that $f|_D$ is a bijection from D to $[-1,\infty)$. Justify your answer.

(1) Since $f(0)=f(2)=0$, f is not one-to-one.

(2) Take $D=[1,\infty)$. To show that $g=f|_D$ is a bijection from $[1,\infty)$ to $[-1,\infty)$, we need to

(i) Show that $\mathrm{Rng}(g) = [-1, \infty)$. Since we already know $\mathrm{Rng}(g) \subseteq \mathrm{Rng}(f) = [-1, \infty)$, we only need to show that $[-1, \infty) \subseteq \mathrm{Rng}(g)$.
Let $y \in [-1, \infty)$. Take $x = \sqrt{y+1} + 1$, then x is well-defined and $x \in [1, \infty)$, and $g(x) = (x-1)^2 - 1 = (y+1) - 1 = y$. Therefore $y \in \mathrm{Rng}(g)$. Therefore $[-1, \infty) \subseteq \mathrm{Rng}(g)$.

(ii) Show that g is one-to-one. Let x_1 and x_2 be elements in $[1, \infty)$ with $g(x_1) = g(x_2)$, that is, $(x_1 - 1)^2 - 1 = (x_2 - 1)^2 - 1$. Then $(x_1 - 1)^2 = (x_2 - 1)^2$. Since $x_1 - 1 \geq 0$, $x_2 - 1 \geq 0$, we get $x_1 - 1 = x_2 - 1$, $x_1 = x_2$. Therefore g is one-to-one.

Problem 3. (20 points) Use the definition of limit to prove that

$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$$

Let $\epsilon > 0$. Take $\delta = \min\{1, \epsilon\} > 0$, then for any x with $0 < |x - 2| < \delta$, we have $x > 2 - \delta \geq 1$ and thus x is positive. Then

$$|\frac{1}{x} - \frac{1}{2}| = |\frac{2-x}{2x}| = \frac{1}{2x} \cdot |2 - x| < \frac{1}{2 \cdot 1}\delta \leq \frac{\epsilon}{2} < \epsilon$$

where we used $x \geq 1$ and $|2 - x| < \delta$ in the first inequality, and $\delta \leq \epsilon$ in the second inequality. Therefore $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

Problem 4. (20=5+5+5+5 points) Determine whether the following limits of sequences exist. If yes, find it. You only need to give an answer, but partial credits could be given to intermediate results.

$$(1): \quad \lim_{n \to \infty} \frac{2n+3}{n^2 - 5} = 0$$

$$(2): \quad \lim_{n \to \infty} \frac{2n^2(-1)^n + 3}{n^2 - 5}$$

does not exist, because x_{2n} converges to 2 and x_{2n+1} converges to -2.

$$(3): \quad \lim_{n \to \infty} \frac{2n^2 + 3(-1)^n}{n^2 - 5} = 2$$

by Sandwich Theorem with $a_n = \frac{2n^2-3}{n^2-5}$, $c_n = \frac{2n^2+3}{n^2-5}$, both converging to 2.

$$(4): \quad \lim_{n \to \infty} x_n, \quad \text{where } x_n \text{ is defined by } x_1 = 2 \text{ and } x_n = \frac{n^2 - 1}{n^2} x_{n-1} \text{ for } n \geq 2.$$

By induction, one can show that $x_n = \frac{n+1}{n}$ (you could observe this by looking at a few terms). Therefore $\lim_{n \to \infty} x_n = 1$.

Problem 5. (10 points) Let A be a nonempty subset of $\mathbb{R}$, and assume $\sup A = s$. Prove that there exists a sequence $\{x_n\}$ with $x_n \in A$ for each $n \in \mathbb{N}$, and $\lim_{n \to \infty} x_n = s$.

We know that for any $\epsilon > 0$, there exists $x \in A$ such that $x > s - \epsilon$.
For any $n \in \mathbb{N}$, applying this statement with $\epsilon = \frac{1}{n}$, and denote the corresponding x as x_n.
Then we get a sequence $\{x_n\}$ with $x_n \in A$ and $x_n > s - \frac{1}{n}$.
We claim that for this sequence $\{x_n\}$, $\lim_{n\to\infty} x_n = s$. In fact, since $x_n \in A$ and $\sup A = s$, we have $x_n \leq s$ for any n. Also, $x_n > s - \frac{1}{n}$, and then we have

$$s - \frac{1}{n} \leq x_n \leq s, \quad \forall n \in \mathbb{N}$$

Then we get $\lim_{n\to\infty} x_n = s$ by Sandwich Theorem.