MATH 461 — Quiz 8 solution

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Problem 1. (1) (7 points) Diagonalize the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}$$

(that is, find invertible P and diagonal D such that $P^{-1}AP = D$).
(2) (3 points) Use the result of (1) to compute A^{100}.

(1)

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

Therefore the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$.
For $\lambda_1 = -1$,

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Therefore an eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.
For $\lambda_1 = 1$,

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Therefore an eigenvector is $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
Therefore $P^{-1}AP = D$ with

$$P = \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2) Since $A = PDP^{-1}$, we have

$$A^{100} = PD^{100}P^{-1} = PIP^{-1} = I$$

since $D^{100} = \begin{pmatrix} (-1)^{100} & 0 \\ 0 & 1^{100} \end{pmatrix} = I$.