

Problem 1. Compute the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix}$$

and find an eigenvector corresponding to each eigenvalue.

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 2 & 2-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2 - 6) = (2-\lambda)(\lambda-4)(\lambda+1)$$

Therefore we have 3 eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = -1$.

For $\lambda_1 = 2$,

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 3 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

One eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 4$,

$$\left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 2 & -2 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

One eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda_3 = -1$,

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

One eigenvector is $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -3/2 \\ 1 \end{pmatrix}$.