MATH 461 — Midterm Exam 2 Solution

30 Mar 2021, Instructor: Ruiwen Shu

Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use MATLAB or the internet in this exam.

Problem 1. (20=10+10 points) Determine whether the following matrices are invertible. If yes, compute its inverse.

(1)

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 3 \\ -2 & 3 & -3 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 4 & 1 & -1 & 10 & 9 \\ 3 & 2 & 0 & 20 & -5 \\ 7 & 3 & 3 & 30 & 2 \\ 2 & 4 & 1 & 40 & 3 \\ 4 & 5 & 3 & 50 & -9 \end{pmatrix}$$

(1) Use elementary row operations:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 0 & 3 & 0 & 1 & 0 \\ -2 & 3 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & -6 & 5 & -2 & 1 & 0 \\ 0 & 6 & -4 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & -6 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -5/6 & 2/6 & -1/6 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & -3/6 & 4/6 & 5/6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & -3/4 \\ 0 & 1 & 0 & -3/6 & 4/6 & 5/6 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

Therefore the desired inverse matrix is

$$\begin{pmatrix} 3/4 & -1/2 & -3/4 \\ -3/6 & 4/6 & 5/6 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & -1/2 & -3/4 \\ -1/2 & 2/3 & 5/6 \\ -1 & 1 & 1 \end{pmatrix}$$

(2) This matrix is not invertible because the second column and the fourth column are linearly dependent.

Problem 2. (20=10+10 points) Compute the determinant of the following matrices:
(1)

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 7 & 3 & 1 \\ 0 & -2 & 1 & 4 \\ -1 & 1 & 0 & 5 \end{pmatrix}$$

(2) The matrix AB, where

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 3 & -4 & 0 \\ -6 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 7 & -18 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

(1) Use elementary row operations,

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 4 & 7 & 3 & 1 \\ 0 & -2 & 1 & 4 \\ -1 & 1 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 7 & -11 \\ 0 & -2 & 1 & 4 \\ 0 & 3 & -1 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 7 & -11 \\ 0 & 0 & -13 & 26 \\ 0 & 0 & 20 & -25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 7 & -11 \\ 0 & 0 & -13 & 26 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

Therefore its determinant is $1 \cdot (-1) \cdot (-13) \cdot 15 = 195$.

(2) $\det(AB) = \det(A) \cdot \det(B)$. A and B are triangular matrices, and therefore $\det(A) = 3 \cdot (-4) \cdot 5 = -60$, $\det(B) = 2 \cdot 1 \cdot 5 = 10$. Therefore $\det(AB) = -600$.

Problem 3. (20 points) Determine whether each of the following sets is a subspace of $\mathbb{R}^3$. Justify your answer in either case.

(1)

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 + 4x_2 - x_3 = 0 \text{ and } x_1 - 9x_2 + x_3 = 0 \right\}$$

(2)

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 + 4x_2 - x_3 = 0 \text{ or } x_1 - 9x_2 + x_3 = 0 \right\}$$

(1) V is a subspace. To prove this, we need

- If $\mathbf{x}, \mathbf{y} \in V$, then $2x_1 + 4x_2 - x_3 = 0$ and $x_1 - 9x_2 + x_3 = 0$ and $2y_1 + 4y_2 - y_3 = 0$ and $y_1 - 9y_2 + y_3 = 0$. Therefore $2(x_1+y_1) + 4(x_2+y_2) - (x_3+y_3) = 0$ and $(x_1+y_1) - 9(x_2+y_2) + (x_3+y_3) = 0$. Therefore $\mathbf{x} + \mathbf{y} \in V$.
- If $\mathbf{x} \in V$, $c \in \mathbb{R}$, then $2x_1+4x_2-x_3 = 0$ and $x_1-9x_2+x_3 = 0$. Therefore $2cx_1+4cx_2-cx_3 = 0$ and $cx_1 - 9cx_2 + cx_3 = 0$. Therefore $c\mathbf{x} \in V$.
- Clearly $x_1 = x_2 = x_3 = 0$ satisfies $2x_1 + 4x_2 - x_3 = 0$ and $x_1 - 9x_2 + x_3 = 0$. Therefore $\mathbf{0} \in V$.

Therefore V is a subspace.

(2) W is not a subspace, because $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in W$ (since it satisfies the first equation), $\mathbf{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in W$ (since it satisfies the second equation), but $\mathbf{x} + \mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \notin W$ (since it satisfies neither equation).

Problem 4. (10 points) Determine whether each of the following sets is a subspace of $\mathbb{R}^3$. Justify your answer in either case.

(1)

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0 \text{ and } x_2 \geq 0 \right\}$$

(2)

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 2x_1 + 4x_2 - x_3 = 10 \right\}$$

(1) V is not a subspace, because $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in V$ but $(-1)\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \notin V$.

(2) W is not a subspace because $\mathbf{0} \notin W$.

Problem 5. (30=15+15 points) Let P_n be the vector space of polynomials of degree no more than n. (You are allowed to use the isomorphism between P_n and $\mathbb{R}^{n+1}$.)

(1) Show that $B = \{1, t+1, t^2 - t + 2, t^3 - t + 3\}$ is a basis of P_3.

(2) Let $p(t) = (1+t)^3$. Compute the coordinate vector $[p]_B$.

(1) Let C be the basis $\{1, t, t^2, t^3\}$ of P_3. We know that the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_C$ is an isomorphism between P_3 and $\mathbb{R}^4$. Therefore, to show that B is a basis of P_3, it suffices to show that the image of B under this isomorphism,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^4$. This is clear since the 4×4 matrix with these vectors as columns is an upper triangular matrix with $\det = 1$.

(2) Using the above isomorphism, the image of $p(t) = 1 + 3t + 3t^2 + t^3$ is

$$\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

and it suffices to obtain the coordinate vector of this vector associated to the above basis of $\mathbb{R}^4$. This can be done by solving the linear system with the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

Use elementary row operations,

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore

$$[p]_B = \begin{pmatrix} -15 \\ 7 \\ 3 \\ 1 \end{pmatrix}$$