MATH 310 — MIDTERM EXAM 2 SOLUTION

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Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use the internet in this exam.
(5) When proving a proposition, you may use conclusions which are significantly simpler than the current proposition.

Problem 1. (20 points) Use induction to prove: for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i \cdot 2^i = (n-1)2^{n+1} + 2$$

STEP 1: Since $\sum_{i=1}^{1} i \cdot 2^i = 1 \cdot 2^1 = 2 = (1-1)2^{1+1} + 2$, the statement is true for 1.
STEP 2: Assume the statement is true for $n \in \mathbb{N}$, that is, $\sum_{i=1}^{n} i \cdot 2^i = (n-1)2^{n+1} + 2$. Then

$$\sum_{i=1}^{n+1} i \cdot 2^i = \sum_{i=1}^{n} i \cdot 2^i + (n+1)2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1} = (2n)2^{n+1} + 2 = ((n+1)-1)2^{(n+1)+1} + 2$$

Therefore the statement is also true for $n+1$.
Therefore, by PMI, the statement is true for any $n \in \mathbb{N}$.

Problem 2. (20=15+5 points) Let

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \sin x - \sin y = 0\}$$

(1) Prove that R is an equivalence relation.
(2) Enumerate 3 distinct elements in the equivalence class $\bar{1}$. (no need to prove)

(1) Reflexive: Let $x \in \mathbb{R}$. Then $\sin x - \sin x = 0$. Therefore xRx.
Symmetric: Let $x, y \in \mathbb{R}$. Suppose xRy. Then $\sin x - \sin y = 0$. Then $\sin y - \sin x = 0$. Therefore yRx.
Transitive: Let $x, y, z \in \mathbb{R}$. Suppose xRy and yRz. Then $\sin x - \sin y = 0$ and $\sin y - \sin z = 0$. Therefore $\sin x = \sin y = \sin z$. Therefore $\sin x - \sin z = 0$. Therefore xRz.

(2) For example, $1, 1 + 2\pi, 1 + 4\pi$. (In fact, $\bar{1} = \{1 + 2k\pi : k \in \mathbb{Z}\} \cup \{(\pi - 1) + 2k\pi : k \in \mathbb{Z}\}$.)

Problem 3. (25=15+10 points) Let

$$R = \{((x, y), (z, w)) \in \mathbb{N}^2 \times \mathbb{N}^2 : x = z \text{ and } y \leq w\}$$

(1) Prove that R is a partial ordering.
(2) For the following sets, determine whether its supremum exists. If yes, find it. (You only need to give an answer.)

$$\text{(i): } \{(2, 3), (2, 6), (2, 5)\}, \quad \text{(ii): } \{(1, 3), (3, 3), (5, 3)\}$$

(1) Reflexive: Let $(x, y) \in \mathbb{N}^2$. Then $x = x$ and $y \leq y$. Therefore $(x, y)R(x, y)$.
Antisymmetric: Let $(x, y), (z, w) \in \mathbb{N}^2$. Suppose $(x, y)R(z, w)$ and $(z, w)R(x, y)$. Then $x = z$, $y \leq w$, $z = x$, $w \leq y$. Therefore $x = z$ and $y = w$, that is, $(x, y) = (z, w)$.
Transitive: Let $(x, y), (z, w), (s, t) \in \mathbb{N}^2$. Suppose $(x, y)R(z, w)$ and $(z, w)R(s, t)$. Then $x = z$, $y \leq w$, $z = s$, $w \leq t$. Therefore $x = s$ and $y \leq t$, that is, $(x, y)R(s, t)$.

(2) (i) Yes, $(2, 6)$.
(ii) No. (because if (x, y) was an upper bound of $\{(1, 3), (3, 3), (5, 3)\}$, then $(1, 3)R(x, y)$, $(3, 3)R(x, y)$ which implies $x = 1$ and $x = 3$, a contradiction.)

Problem 4. (10=5+5 points) Let

$$f(x) = x^3 + x^2, \quad g(x) = e^x + 2x$$

be real functions. Compute $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x)) = f(e^x + 2x) = (e^x + 2x)^3 + (e^x + 2x)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^3 + x^2) = e^{x^3+x^2} + 2(x^3 + x^2)$$

Problem 5. (25=15+10 points) Let

$$f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - \frac{y}{2} + 3 = 0\}$$

(1) Prove that f is a function from $\mathbb{Z}$ to $\mathbb{Z}$.
(2) Prove that the range of f is the set of all even integers.

(1) STEP 1: prove $\text{Dom}(f) = \mathbb{Z}$. It suffices to prove $\mathbb{Z} \subseteq \text{Dom}(f)$.
Let $x \in \mathbb{Z}$. Take $y = 2x + 6 \in \mathbb{Z}$, and then $x - \frac{y}{2} + 3 = x - \frac{2x+6}{2} + 3 = 0$, that is, $(x, y) \in f$. Therefore $x \in \text{Dom}(f)$.

STEP 2: prove: if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.
Let $x, y, z \in \mathbb{Z}$. Assume $(x, y) \in f$ and $(x, z) \in f$. Then $x - \frac{y}{2} + 3 = 0$ and $x - \frac{z}{2} + 3 = 0$. Then $\frac{y}{2} = x + 3 = \frac{z}{2}$. Therefore $y = z$.

(2) Part (1) STEP 1 already gives $f(x) = 2x + 6$ for any $x \in \mathbb{Z}$.
STEP 1: prove $\text{Rng}(f) \subseteq \{\text{even integers}\}$.
Let $y \in \text{Rng}(f)$. Then there exists $x \in \mathbb{Z}$ such that $y = f(x) = 2x + 6 = 2(x + 3)$. Therefore y is even.
STEP 2: prove $\{\text{even integers}\} \subseteq \text{Rng}(f)$.
Let y be an even integer. Then there exists $k \in \mathbb{Z}$ such that $y = 2k$. Take $x = k - 3 \in \mathbb{Z}$, and then $f(x) = 2(k - 3) + 6 = 2k = y$. Therefore $y \in \text{Rng}(f)$.