Extending Justin's Guide to MATLAB in MATH 240/461 - Part 4

New Commands

1. Eigenvalues can be found easily. If `A` is a matrix then:

   ```
   >> eig(A)
   ```

   will return the eigenvalues. Note that it will return complex eigenvalues too. So keep an i open for those.

2. If we have an eigenvalue λ for A, we can use `rref` on an augmented matrix $[A - \lambda I \mid \mathbf{0}]$ to lead us to the eigenvectors. For example if A is 4×4 and $\lambda = 3$ is an eigenvalue, then we can obtain the coefficient matrix of this system by entering `A - 3*eye(4)`.

3. Even better: MATLAB can do everything in one go. If you recall from class, *diagonalizing* a matrix A means finding a diagonal matrix D and an invertible matrix P with $A = PDP^{-1}$. The diagonal matrix D contains the eigenvalues along the diagonal and the matrix P contains eigenvectors as columns, with column j of P corresponding to the eigenvalue in column j of D.

   To do this we use the `eig` command again but demand different output. The format is:

   ```
   >> [P,D]=eig(A)
   ```

   which assigns `P` and `D` for A, if possible. If it's not possible MATLAB returns very strange-looking output.

4. We can compute the dot product of two vectors using the command `dot`. For example:

   ```
   >> dot([1;2;4],[-2;1;5])
   ```

5. We can find the length of a vector from the basic definition. If `v` is a vector then:

   ```
   >> sqrt(dot(v,v))
   ```

6. Or we can just use the `norm` command:

   ```
   >> norm(v)
   ```

7. To get the transpose of a matrix `A` we do:

   ```
   >> transpose(A)
   ```

   or

   ```
   >> A'
   ```

   Note that `A'` actually returns that *conjugate transpose* (or *adjoint*) of `A`. That is, complex conjugation is also applied to all of the entries of A^T. This doesn't make a difference if the entries of the matrix are real numbers, but it will make a difference if they are (nonreal) complex numbers

MATH 461 Spring 2021 MATLAB Project 4 – due Wed, 4/28 by 11:59 pm

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- **Directions:** The same directions from the previous MATLAB projects apply here. Please reread them. You will need to write a script (.m file) and then "publish" it in MATLAB as a pdf in order to turn it in on Gradescope.
- **Format:** For this project, do the first problem in `format rat` and do the rest in in `format short`.

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1. (Use `format rat`) Recall $[\mathbf{x}]_{\mathcal{B}}$ denotes the coordinate vector of $\mathbf{x}$ with respect to a basis $\mathcal{B}$ for a vector space V. Given two bases $\mathcal{B}$ and $\mathcal{C}$ for V, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ denotes the change of coordinates matrix, which has the property that
$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}} \quad \text{for all } \mathbf{x} \in V.$$
It follows that
$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}.$$
Also, if we have three bases $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$, then
$$\left(\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\right)\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right) = \underset{\mathcal{D}\leftarrow\mathcal{B}}{P}.$$
Each of the following three sets is a basis for the vector space $\mathbb{P}_3$:
$$\begin{aligned} &\mathcal{E} = \{1,\ t,\ t^2,\ t^3\}\ \ , \\ &\mathcal{B} = \{1,\ 1+2t,\ 2-t+3t^2,\ 4-2t+t^3\}\ \ , \quad \text{and} \\ &\mathcal{C} = \{1+3t+t^3,\ 2+t,\ 3t-t^2+4t^3,\ 3t\}\ \ . \end{aligned}$$
   (a) Find and enter the matrices $P = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ and $Q = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}$.

   (b) Use P and Q and the properties above to compute $R = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

   (c) Compute the $\mathcal{C}$ coordinate vector of the polynomial t^3.

   (d) Suppose $p(t)$ is the polynomial for which $[p(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. Compute the coordinate vector $[p(t)]_{\mathcal{C}}$.

   (e) Let $p(t)$ denote the polynomial from the previous part. Express this polynomial in the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_2 t^3$.

2. (Use `format short` from here onward) Let's observe the destructive nature of elementary row operations on the eigenvalues of a matrix. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$.

   (a) Use the command `eig(A)` to find the eigenvalues of A.

   (b) Consider the following matrices obtained from A by elementary row operations:

      i. B is obtained from A by swapping row 1 with row 2.

      ii. C is obtained from A by multiplying row 1 by 7.

      iii. D is obtained from A by adding 3 times row 1 from row 3.

   Use MATLAB to find the eigenvalues of B, C,and D.

   (c) ⋆ The eigenvalues of these matrices were different, and they changed in unpredictable ways. However, they each had one eigenvalue in common. Explain why this eigenvalue will remain no matter what row operation is performed. (Hint: there is a relevant theorem in §5.2)

3. Let $A = \begin{bmatrix} 7 & 6 & -1 \\ -8 & -7 & 1 \\ 4 & 2 & -1 \end{bmatrix}$.

   (a) Execute the command `[P,D] = eig(A)` to diagonalize A.

   (b) Use MATLAB to verify that $A = PDP^{-1}$.

   (c) ⋆ Use the previous results to give the eigenvalues of A, and give an eigenvector for each eigenvalue.

4. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

   (a) Execute the command `[P,D]=eig(A)`. Something strange should occur in the output (take a close look at P).

   (b) Use MATLAB to try see if $A = PDP^{-1}$.

   (c) Find a basis for the eigenspace of A corresponding to the eigenvalue $\lambda = 2$.

   (d) ⋆ Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors for A? Does this explain why something went wrong in part (b)? (There is a relevant theorem in §5.3.)

5. Let $A = \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & 0 \\ 0 & 2 & 5 \\ 3 & 1 & 3 \end{bmatrix}$.

   (a) Enter A into MATLAB and use MATLAB to compute the dot product of the first column of A with its second column. Also compute the dot product of the third column with itself. (See guide for how to extract a column from a matrix.)

   (b) Compute the matrix product $A^T A$.

   (c) ⋆ What is the relationship between the entries of $A^T A$ and the dot products of the columns of A? Make sure your answer is consistent with your computations.

   (d) ⋆ What in general is the relationship between the entries of AA^T and dot products of vectors associated to A?

   (e) Compute AA^T and do at least two dot product computations to support your previous answer.

   (f) Let $Q = \begin{bmatrix} 1/\sqrt{14} & 1/\sqrt{3} & 5/\sqrt{42} \\ 2/\sqrt{14} & 1/\sqrt{3} & -4/\sqrt{42} \\ 3/\sqrt{14} & -1/\sqrt{3} & 1/\sqrt{42} \end{bmatrix}$. Do a single matrix computation which shows that the columns of form an orthonormal set.

   (g) ⋆ Explain carefully why your computation above shows that the columns form an orthonormal set.

   (h) ⋆ Explain why the rows of Q must also form an orthonormal set.