MATH 461 Spring 2021 MATLAB Project 3

- **Directions:** The same directions from the previous MATLAB projects apply here. Please reread them. You will need to write a script (.m file) and then "publish" it in MATLAB as a pdf in order to turn it in on Gradescope.
- There are no new commands to learn for this project. Please review old commands as needed.
- **Format:** For this project, do problem 2 in `format short` and do the rest in `format rat`.

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1. Do #40 on p. 239 of the textbook. (If you have the 5th edition, it is #36 on p. 226.)
2. (Use `format short`) Consider the set of functions
$$\left\{1, \cos t, \cos^2 t, \cos^3 t\right\}.$$
We can view this set as a set of vectors in the vector space $C(\mathbb{R})$ of all continuous real-valued functions whose domain is $\mathbb{R}$. We would like to show that this is a linearly independent set in $C(\mathbb{R})$. This means that if x_1, x_2, x_3, x_4 are scalars such that
$$(*) \qquad x_1(1) + x_2 \cos t + x_3 \cos^2 t + x_4 \cos^3 t = 0 \qquad \text{(for all } t\text{)},$$
then we must have $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$. Essentially we are trying to disprove the existence of a linear relation relating these four functions to each other. More concretely, we are trying to prove that no identity such as
$$3 + 4\cos t - 7\cos^2 t + \frac{1}{2}\cos^3 t = 0$$
could possibly exist.
   (a) Each substitution of a number for t in the equation $(*)$ produces a linear equation in the four variables x_1, x_2, x_3, x_4. By plugging in $t = 0, 0.1, 0.2$ and 0.3, you get four linear equations for the four unknowns. Define the coefficient matrix A for this linear system in MATLAB.
   (b) Note that a nontrivial solution $\mathbf{x}$ to $(*)$ is automatically a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. However, if A is invertible, then $A\mathbf{x} = \mathbf{0}$ has no nontrivial solutions. This implies that the equation $(*)$ has no nontrivial solutions. Compute `rref(A)` and `det(A)`.
   (c) $\star$ Very briefly explain why each of the last two computations show A is invertible.
   (d) It is reasonable to be suspicious of the very small value of $\det(A)$ in the last step – could this be roundoff error, with the actual $\det(A)$ being zero? Do a check by repeating the computation with the more spread out inputs $t = 0, .2, .5, 1$, to see $\det(A)$ large enough to eliminate that suspicion.
   (e) $\star$ Now consider the set of functions $\{1, \sin^2 t, \cos^2 t\}$. (Here 1 denotes the constant function whose value is always 1.) Explain why this set of functions is linearly dependent. (Hint: Do you know any identities that relate these three functions?)
   (f) **(Optional - don't have to turn in)** Explore what happens if you try to use the techniques of the previous parts on the set $\{1, \sin^2 t, \cos^2 t\}$.
3. Do #54 on p. 249 of the textbook. (If you have the 5th edition, it is #34 on p. 232.)
4. (Use `format rat`) Let $A = \begin{bmatrix} 0 & 1 & 1 & 9 & 1 \\ -9 & -8 & 13 & 9 & -2 \\ -3 & -3 & 4 & 0 & -1 \\ -6 & -4 & 10 & 18 & 0 \end{bmatrix}$.
   (a) Compute $\operatorname{rank} A$. (The MATLAB command is `rank(A)`.)
   (b) $\star$ Use the rank to determine the values $\dim(\operatorname{Nul} A), \dim(\operatorname{Col} A)$.

(c) Compute `rref(A)` and use it to give a basis for
   i. Nul A
   ii. Col A

5. Consider the polynomials

$$p_1(t) = 1 + t + 2t^2 + 3t^3, \quad p_2(t) = -3 + 5t + t^2 + 8t^3, \quad p_3(t) = 5 + 3t + 5t^3,$$
$$p_4(t) = 1 + 8t + 4t^2 + 15t^3, \quad p_5(t) = 2 + t - t^2 + t^3,$$

which are all elements of the vector space $\mathbb{P}_3$. We shall investigate the subspace

$$W = \text{Span}\{p_1(t),\ p_2(t),\ p_3(t),\ p_4(t),\ p_5(t)\}.$$

(a) Let $\mathbf{v}_i = [p_i(t)]_{\mathcal{E}}$, the coordinate vector of $p_i(t)$ relative to the basis $\mathcal{E} = \{1, t, t^2, t^3\}$ for $\mathbb{P}_3$. Enter these coordinate vectors into MATLAB as `v1, v2, v3, v4, v5`.

(b) Let A be the matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5\end{bmatrix}$. Observe that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} = \text{Col}(A)$. Use this fact to compute a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. (Recall you can enter A into MATLAB as `A = [v1 v2 v3 v4 v5]`.)

(c) $\star$ Translate your previous answer into a basis for W (consisting of polynomials). What is $\dim W$?

(d) $\star$ Is $W = \mathbb{P}_3$? Justify your answer.

6. Consider the following four matrices from the vector space $M_{2\times 3}$ of all 2×3 matrices:

$$A_1 = \begin{bmatrix} -3 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -5 & -2 & 2 \\ -4 & 1 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -2 & 2 & 1 \\ 4 & 2 & -3 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -2 & -12 \end{bmatrix}.$$

(a) Let $\mathbf{v}_i$ denote the coordinate vector $[A_i]_{\mathcal{E}}$ relative to the basis

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Enter the coordinate vectors for A_1, A_2, A_3, A_4 into MATLAB.

(b) Use MATLAB to show that the coordinate vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

(c) $\star$ Express one of the matrices A_i as a linear combination of the other three. (Hint: first do the same for the coordinate vectors.)