23 Feb 2021, Instructor: Ruiwen Shu

Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use MATLAB or the internet in this exam.

Problem 1. (30=15+15 points) Solve the following linear systems, and express your solution into parametric form, if there is any solution(s).

(1)
$$\begin{cases} 2x_1 - x_2 - x_3 = 2 \\ x_1 + 3x_2 + 2x_3 = 2 \end{cases}$$

(2)
$$\begin{cases} x_2 + x_3 = 3 \\ x_1 - 3x_2 + 2x_3 = 2 \\ 2x_1 - 5x_2 + 5x_3 = 1 \end{cases}$$

(1) The augmented matrix is
$$\begin{pmatrix} 2 & -1 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$
Use elementary row operations to make it into reduced echelon form:
$$\begin{pmatrix} 2 & -1 & -1 & 2 \\ 0 & \frac{7}{2} & \frac{5}{2} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -1 & -1 & 2 \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}$$
$$\begin{pmatrix} 2 & 0 & -\frac{2}{7} & \frac{16}{7} \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{7} & \frac{8}{7} \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} \end{pmatrix}$$
Therefore the solution is
$$x_1 = \frac{8}{7} + \frac{1}{7}x_3, \quad x_2 = \frac{2}{7} - \frac{5}{7}x_3, \quad x_3 \text{ free}$$
Write into parametric form,
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{8}{7} + \frac{1}{7}x_3 \\ \frac{2}{7} - \frac{5}{7}x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{8}{7} \\ \frac{2}{7} \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \frac{1}{7} \\ -\frac{5}{7} \\ 1 \end{pmatrix}$$

(2) The augmented matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & -3 & 2 & 2 \\ 2 & -5 & 5 & 1 \end{pmatrix}$$

Use elementary row operations to make it into echelon form:

$$\begin{pmatrix} 1 & -3 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 2 & -5 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

and the last row gives a contradiction. Therefore the linear system has no solution.

Problem 2. (20=10+10 points) Determine whether the following sets of vectors are linearly dependent. Your answer should be justified.

(1) $\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -7 \end{pmatrix}.$

(2) $\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ -6 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}.$

(1) Write the matrix with columns being the given vectors,

$$\begin{pmatrix} 0 & 2 & -2 \\ -1 & -1 & 3 \\ -2 & 3 & -7 \end{pmatrix}$$

Use elementary row operations to make it into echelon form:

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & 2 & -2 \\ -2 & 3 & -7 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 5 & -13 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{pmatrix}$$

Every column has a pivot. Therefore the given vectors are linearly independent.

(2) The number of vectors is 4 and the length of vectors is 3, $4 > 3$, therefore the given vectors are linearly dependent.

Problem 3. (20=4+8+8 points) Let T be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 4 & k \end{pmatrix}$$

Here $k \in \mathbb{R}$ is a constant.

(1) What are the domain and codomain of T?
(2) Find all values of k such that T is one-to-one.
(3) Find all values of k such that T is onto.

(1) The domain is $\mathbb{R}^3$. The codomain is $\mathbb{R}^2$.

(2) Since the dimension of the domain is greater than that of the codomain, T is never one-to-one.

(3) Use elementary row operations to make A into echelon form:

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & k+4 \end{pmatrix}$$

To make T onto, we need every row to have a pivot. Therefore T is onto when $k \neq -4$.

Problem 4. (20=5+5+5+5 points) Let

$$A = \begin{pmatrix} 3 & -2 & -3 \\ 1 & 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$$

Determine whether the following expressions are well-defined. If yes, compute it. If no, explain why not.

(1) $A - B + 2C$.
(2) $2C + 3B$.
(3) AB.
(4) BA.

(1) Not well-defined, because A and B have different dimensions.

(2)

$$2C + 3B = \begin{pmatrix} -6 & 2 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} 6 & 3 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 7 & 4 \end{pmatrix}$$

(3) Not well-defined, because A is 2×3, B is 2×2, and their inner dimensions (3 and 2) are not equal.

(4)

$$BA = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 & -3 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 1 & 2 \cdot (-2) + 1 \cdot 2 & 2 \cdot (-3) + 1 \cdot 5 \\ 1 \cdot 3 + 4 \cdot 1 & 1 \cdot (-2) + 4 \cdot 2 & 1 \cdot (-3) + 4 \cdot 5 \end{pmatrix} = \begin{pmatrix} 7 & -2 & -1 \\ 7 & 6 & 17 \end{pmatrix}$$

Problem 5. (10=5+5 points) Let

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

(1) Find a 2×2 matrix B such that $B \neq 0$ but $AB = 0$.

(2) Find a 2×2 matrix B such that $B \neq 0$ but $BA = 0$.

(The answer to this question is not unique.)

(1)

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

(2)

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$