MATH 310 Midterm Exam 1 Solution

23 Feb 2021, Instructor: Ruiwen Shu

Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use the internet in this exam.
(5) When proving a proposition, you may use conclusions which are significantly simpler than the current proposition.

The solutions to the proof problems may not be unique.

Problem 1. (10=5+5 points) Let P, Q be propositions, such that P is true and Q is false. Determine the truth values of the following propositions. Show your work.

(1) $(\neg P) \vee Q$.
(2) $(P \Rightarrow Q) \wedge (P \vee Q)$.

(1) $\neg P$ is false and Q is false. Therefore $(\neg P) \vee Q$ is false.

(2) $P \Rightarrow Q$ is false. Therefore $(P \Rightarrow Q) \wedge (P \vee Q)$ is false.

Problem 2. (30=15+15 points) Prove:

(1) Let n be an integer. Then $n^2 + 3n$ is even.

(2) Let x be a real number. If $x^2 + \sqrt{x} \leq 2$, then $x \leq 1$.

(1) Let n be an integer. Then either n is even or n is odd.
Case 1: If n is even, then $n + 3$ is odd. Therefore $n^2 + 3n = n(n + 3)$ is even.
Case 2: If n is odd, then $n + 3$ is even. Therefore $n^2 + 3n = n(n + 3)$ is even.
Therefore $n^2 + 3n$ is even.

(2) Let x be a real number. Assume $x > 1$. Then $x^2 > 1$, $\sqrt{x} > 1$. Therefore $x^2 + \sqrt{x} > 2$. Therefore $x > 1$ implies $x^2 + \sqrt{x} > 2$. Therefore by contraposition, $x^2 + \sqrt{x} \leq 2$ implies $x \leq 1$.

Problem 3. (20 points) Prove: There exists a real number x, such that for any real number

y, there exists a real number z such that

$$z^2 + \sin y = x$$

Take $x = 1$. Then for any real number y, we have $x - \sin y \geq 1 - 1 \geq 0$ since $\sin y \leq 1$. Then we may take $z = \sqrt{x - \sin y}$, which satisfies $z^2 = x - \sin y$, that is, $z^2 + \sin y = x$.

Problem 4. (20=5+5+5+5 points) Compute the following sets. Express your final result by: either enumerate all the elements in the set, or express as a union of disjoint intervals. Show your work.

(1) $(A \cup B) \cap C$, where $A = [0, 2]$, $B = (3, 6)$, $C = [2, 3]$ are intervals.
(2) $A^c \cup B$, where $A = [0, 2]$, $B = (-1, 4)$ are intervals.
(3) $\mathscr{P}(A) - \mathscr{P}(B)$, where $A = \{1, 2, 3\}$, $B = \{3, 5\}$, and $\mathscr{P}$ denotes the power set.
(4) $(A \times B) \cap (B \times A)$, where $A = \{0, 1, 2\}$, $B = \{2, 3\}$.

(1) $A \cup B = [0, 2] \cup (3, 6)$. Therefore $(A \cup B) \cap C = \{2\}$.

(2) $A^c = (-\infty, 0) \cup (2, \infty)$. Therefore $A^c \cup B = (-\infty, \infty) = \mathbb{R}$.

(3) $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, $\mathscr{P}(B) = \{\emptyset, \{3\}, \{5\}, \{3, 5\}\}$. Therefore $\mathscr{P}(A) - \mathscr{P}(B) = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

(4) $A \times B = \{(0, 2), (0, 3), (1, 2), (1, 3), (2, 2), (2, 3)\}$, $B \times A = \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$. Therefore $(A \times B) \cap (B \times A) = \{(2, 2)\}$. (Or, you may use $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B) = \{2\} \times \{2\} = \{(2, 2)\}$.

Problem 5. (20 points) Prove: Let

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \quad B = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}, \quad C = \{(x, y) \in \mathbb{R}^2 : |y| \leq 1\}$$

Then A is a proper subset of $B \cap C$.

Part 1: prove $A \subseteq B \cap C$. Let $(x, y) \in A$. Then $x^2 + y^2 \leq 1$. Therefore $x^2 \leq 1 - y^2 \leq 1$ since $y^2 \geq 0$. Therefore $|x| \leq 1$. Therefore $(x, y) \in B$. Similarly one can prove $(x, y) \in C$: $y^2 \leq 1 - x^2 \leq 1$ since $x^2 \geq 0$. Therefore $|y| \leq 1$. Therefore $(x, y) \in C$. Therefore $(x, y) \in B \cap C$. Therefore $A \subseteq B \cap C$.
Part 2: prove $A \neq B \cap C$. Consider the element $(0.9, 0.9)$. Since $0.9^2 + 0.9^2 = 1.62 > 1$, $(0.9, 0.9) \notin A$. However, since $|0.9| \leq 1$, we have $(0.9, 0.9) \in B$ and $(0.9, 0.9) \in C$. Therefore $(0.9, 0.9) \in B \cap C$. Therefore $A \neq B \cap C$.