MATH 246 GROUP WORK 13 SOLUTION
2 DEC 2020, INSTRUCTOR: RUIWEN SHU

Each group only needs to submit ONE file containing your solutions!

Problem 1. Consider the planar system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -e^x(y-1) \\ e^x(x^2-4) \end{pmatrix}$$

(1) Solve the orbit equation (you may leave it as an implicit equation). Is it a Hamiltonian system? Is it a conservative system? (Hint: recall that one can try an integrating factor which is either a function of x or a function of y)

(2) Find all stationary points and determine their type by linearization. Determine whether they are stable/unstable/attracting/repelling. Sketch them in a picture.

(1) The orbit equation is

$$-e^x(x^2-4)\,\mathrm{d}x - e^x(y-1)\,\mathrm{d}y = 0$$

This differential form is not exact, so we look for an integrating factor $\rho(x,y)$ such that

$$-\rho(x,y)e^x(x^2-4)\,\mathrm{d}x - \rho(x,y)e^x(y-1)\,\mathrm{d}y$$

is exact. This requires

$$-\partial_y(\rho(x,y)e^x(x^2-4)) = -\partial_x(\rho(x,y)e^x(y-1))$$

$$\partial_y\rho \cdot e^x(x^2-4) = \partial_x\rho \cdot e^x(y-1) + \rho e^x(y-1)$$

Then notice that setting $\rho = \rho(x)$ (which gives $\partial_y\rho = 0$) could eliminate the y variable and get $\partial_x\rho = -\rho$ which gives a choice $\rho(x) = e^{-x}$. Using this, we get

$$-(x^2-4)\,\mathrm{d}x - (y-1)\,\mathrm{d}y = 0$$

This is a separable equation (a particular case of 'exact'), and can be integrated as

$$-\frac{1}{3}x^3 + 4x - \frac{1}{2}y^2 + y = C$$

which is the solution to the orbit equation. This planar system is conservative but not Hamiltonian, because the orbit equation is not exact but has an integrating factor.
(2) To find stationary points, set

$$-e^x(y-1) = 0, \quad e^x(x^2-4) = 0$$

and get two solutions $(2,1)$ and $(-2,1)$. To do linearization, first compute the Jacobian

$$\partial\mathbf{f} = \begin{pmatrix} -e^x(y-1) & -e^x \\ e^x(x^2-4) + e^x 2x & 0 \end{pmatrix} = e^x\begin{pmatrix} -(y-1) & -1 \\ x^2-4+2x & 0 \end{pmatrix}$$

At $(2,1)$,

$$\partial \mathbf{f}(2,1) = e^2 \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$$

The characteristic polynomial is

$$\det \begin{pmatrix} z & e^2 \\ -4e^2 & z \end{pmatrix} = z^2 + 4e^4$$

whose roots are

$$\lambda_{1,2} = \pm 2e^2 i$$

which are complex with zero real parts. Therefore $(2,1)$ is a center (which is stable). Since $a_{21} > 0$, it is counterclockwise.

At $(-2,1)$,

$$\partial \mathbf{f}(-2,1) = e^2 \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}$$

The characteristic polynomial is

$$\det \begin{pmatrix} z & e^2 \\ 4e^2 & z \end{pmatrix} = z^2 - 4e^4$$

whose roots are

$$\lambda_{1,2} = \pm 2e^2$$

which are real eigenvalues, with one positive and one negative. Therefore $(-2,1)$ is a saddle (which is unstable). To sketch the phase portrait, we also need the eigenvectors. For $\lambda_1 = 2e^2$,

$$e^2 \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2e^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The first component gives

$$-v_2 = 2v_1, \quad v_2 = -2v_1$$

and an eigenvector is

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Similarly an eigenvector for $\lambda_2 = -2e^2$ is

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

See another file for the plot.