Problem 1. Solve the initial-value problem

$$y'' + y' - 2y = 20\sin t, \quad y(0) = -3,\, y'(0) = 1$$

The characteristic polynomial is

$$p(z) = z^2 + z - 2 = (z+2)(z-1)$$

It has two roots $r_1 = -2$, $r_2 = 1$, both with multiplicity 1. Therefore the general solution to homogeneous problem is

$$y_H(t) = C_1 e^{-2t} + C_2 e^t$$

$20\sin t = \frac{20}{2i}(e^{it} - e^{-it})$. For e^{it}, key identity evaluation gives

$$L(e^{it}) = p(i)e^{it} = (-3+i)e^{it}$$

$$L(\frac{20}{2i} \cdot \frac{1}{-3+i} e^{it}) = \frac{20}{2i} e^{it}$$

Taking complex conjugate,

$$L(-\frac{20}{2i} \cdot \frac{1}{-3-i} e^{-it}) = -\frac{20}{2i} e^{-it}$$

Therefore a particular solution is

$$y(t) = \frac{20}{2i} \cdot (\frac{1}{-3+i} e^{it} - \frac{1}{-3-i} e^{-it}) = \frac{20}{2i} \cdot (\frac{-3-i}{10} e^{it} - \frac{-3+i}{10} e^{-it})$$

$$= -6\frac{1}{2i}(e^{it} - e^{-it}) - 2\frac{1}{2}(e^{it} + e^{-it}) = -6\sin t - 2\cos t$$

Then the general solution is

$$y(t) = -6\sin t - 2\cos t + C_1 e^{-2t} + C_2 e^t$$

Matching with initial condition gives

$$-3 = -2 + C_1 + C_2$$

$$1 = -6 - 2C_1 + C_2$$

Solve and get

$$C_1 = -\frac{8}{3}, \quad C_2 = \frac{5}{3}$$

Therefore the solution to IVP is

$$y(t) = -6\sin t - 2\cos t - \frac{8}{3} e^{-2t} + \frac{5}{3} e^t$$

Problem 2. Compute a particular solution to the ODE

$$ty'' + y' = \frac{2}{t}$$

given a fundamental set of solutions to the homogeneous equation $y_1(t) = 1$, $y_2(t) = \ln t$.

Write into standard form

$$y'' + \frac{1}{t}y' = \frac{2}{t^2}$$

We use the variation of parameters. Write

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Then u_1, u_2 satisfies

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

that is,

$$\begin{pmatrix} 1 & \ln t \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{t^2} \end{pmatrix}$$

Solve this linear system and get

$$u_2' = \frac{2}{t}, \quad u_1' = -\frac{2}{t}\ln t$$

Then integrate,

$$u_2 = \int \frac{2}{t}\,\mathrm{d}t = 2\ln t, \quad u_1 = -2\int \frac{\ln t}{t}\,\mathrm{d}t = -(\ln t)^2$$

(the last one by a substitution $v = \ln t$). Therefore a particular solution is

$$y(t) = -(\ln t)^2 + 2(\ln t)^2 = (\ln t)^2$$