MATH 246 MIDTERM EXAM 3 SOLUTION

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Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use MATLAB or the internet in this exam.
(5) You do not need to write the interval of definition unless you are asked to do so.

Problem 1. (20=5+5+5+5 points) Compute: ($\mathcal{L}$ denotes the Laplace transform, $\mathcal{L}^{-1}$ denotes the inverse Laplace transform)

$$(1)\,\mathcal{L}[\cos(2t)];\quad (2)\,\mathcal{L}[t^2e^t(e^t+e^{2t})];\quad (3)\,\mathcal{L}^{-1}[\frac{s}{(s+1)(s+3)}];\quad (4)\,\mathcal{L}^{-1}[\frac{1}{s^3+s}];$$

Express your solution with real-valued functions.

(1) One can apply the formula from textbook $\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$ with $a=0, b=2$ to get $\mathcal{L}[\cos(2t)] = \frac{s}{s^2+4}$. One can also write $\cos(2t) = \frac{1}{2}(e^{2it}+e^{-2it})$ and then $\mathcal{L}[\cos(2t)] = \frac{1}{2}(\frac{1}{s-2i}+\frac{1}{s+2i}) = \frac{s}{s^2+4}$.

(2)

$$\mathcal{L}[t^2e^t(e^t+e^{2t})] = \mathcal{L}[t^2e^{2t}+t^2e^{3t})] = \frac{2}{(s-2)^3}+\frac{2}{(s-3)^3}$$

(3) By partial fractions,

$$\frac{s}{(s+1)(s+3)} = \frac{C_1}{s+1}+\frac{C_2}{s+3},\quad s = C_1(s+3)+C_2(s+1)$$

Evaluating at $s=-1$ gives $C_1 = -\frac{1}{2}$. Evaluating at $s=-3$ gives $C_2 = \frac{3}{2}$. Therefore

$$\mathcal{L}^{-1}[\frac{s}{(s+1)(s+3)}] = -\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t}$$

(4) By partial fractions,

$$\frac{1}{s^3+s} = \frac{1}{s(s^2+1)} = \frac{C_1}{s}+\frac{C_2s+C_3}{s^2+1},\quad 1 = C_1(s^2+1)+(C_2s+C_3)s$$

Compare $s^2, s, 1$ coefficients,

$$0 = C_1+C_2,\quad 0 = C_3,\quad 1 = C_1,\quad \Rightarrow C_2 = -1$$

Therefore

$$\mathcal{L}^{-1}[\frac{1}{s^3+s}] = 1-\cos t$$

Problem 2. (20 points) Solve the initial value problem

$$y' + 2y = u(t-1)e^{3t}, \quad y(0) = 1$$

where u denotes the Heaviside function. Express your solution as a piecewise defined function (u should not appear in your final answer).

Take Laplace transform,

$$s\mathcal{L}[y] - y(0) + 2\mathcal{L}[y] = \mathcal{L}[u(t-1)e^{3(t-1)}e^3] = e^{-s}\frac{1}{s-3}e^3$$

$$(s+2)\mathcal{L}[y] = e^{-s}\frac{1}{s-3}e^3 + 1$$

$$\mathcal{L}[y] = e^{-s}\frac{1}{(s-3)(s+2)}e^3 + \frac{1}{s+2} = e^{-s}(\frac{1}{5}\frac{1}{s-3} - \frac{1}{5}\frac{1}{s+2})e^3 + \frac{1}{s+2}$$

where we omit the procedure of partial fractions. Then take inverse Laplace transform,

$$y(t) = \frac{1}{5}e^3u(t-1)e^{3(t-1)} - \frac{1}{5}e^3u(t-1)e^{-2(t-1)} + e^{-2t} = \begin{cases} e^{-2t}, & 0 \le t < 1 \\ \frac{1}{5}e^3e^{3(t-1)} - \frac{1}{5}e^3e^{-2(t-1)} + e^{-2t}, & t \ge 1 \end{cases}$$

Problem 3. (20=10+10 points) Consider the ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A(t)\mathbf{x}, \quad A(t) = \begin{pmatrix} 2t+1 & -3 \\ 1 & 2t+5 \end{pmatrix}$$

(1) Find the constant c such that

$$\mathbf{x}_1(t) = e^{t^2+ct}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

is a solution.

(2) Given that

$$\mathbf{x}_2(t) = e^{t^2+4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. Is $\mathbf{x}_1, \mathbf{x}_2$ a fundamental set of solutions? Justify you answer.

(1)

$$\frac{\mathrm{d}\mathbf{x_1}}{\mathrm{d}t} = e^{t^2+ct}(2t+c)\begin{pmatrix} -3 \\ 1 \end{pmatrix} = e^{t^2+ct}\begin{pmatrix} -6t-3c \\ 2t+c \end{pmatrix}$$

$$A\mathbf{x}_1 = e^{t^2+ct}\begin{pmatrix} 2t+1 & -3 \\ 1 & 2t+5 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = e^{t^2+ct}\begin{pmatrix} -6t-3-3 \\ -3+2t+5 \end{pmatrix} = e^{t^2+ct}\begin{pmatrix} -6t-6 \\ 2t+2 \end{pmatrix}$$

Matching either component gives $c = 2$.

(2) We only need to compute

$$Wr[\mathbf{x}_1, \mathbf{x}_2](0) = \det \begin{pmatrix} -3e^{t^2+2t} & e^{t^2+4t} \\ e^{t^2+2t} & -e^{t^2+4t} \end{pmatrix} |_{t=0} = \det \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} = 2 \neq 0$$

Therefore $\mathbf{x}_1, \mathbf{x}_2$ is a fundamental set.

Problem 4. (25=15+10 points) Define the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

(1) Find e^{tA}.

(2) Solve the initial value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(1) Method 1: natural fundamental set method.

$$\det(zI - A) = \det \begin{pmatrix} z-3 & -2 \\ -1 & z-2 \end{pmatrix} = z^2 - 5z + 4 = (z-1)(z-4)$$

The roots are $r_1 = 1, r_2 = 4$. The general solution to the ODE $y'' - 5y' + 4y = 0$ is

$$y(t) = C_1 e^t + C_2 e^{4t}, \quad y'(t) = C_1 e^t + 4C_2 e^{4t}$$

For N_0,

$$1 = C_1 + C_2, \quad 0 = C_1 + 4C_2 \quad \Rightarrow C_1 = \frac{4}{3}, C_2 = -\frac{1}{3}$$

$$N_0(t) = \frac{4}{3}e^t - \frac{1}{3}e^{4t}$$

For N_0,

$$0 = C_1 + C_2, \quad 1 = C_1 + 4C_2 \quad \Rightarrow C_1 = -\frac{1}{3}, C_2 = \frac{1}{3}$$

$$N_1(t) = -\frac{1}{3}e^t + \frac{1}{3}e^{4t}$$

Therefore

$$e^{tA} = N_0(t)I + N_1(t)A = (\frac{4}{3}e^t - \frac{1}{3}e^{4t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-\frac{1}{3}e^t + \frac{1}{3}e^{4t}) \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}e^t + \frac{2}{3}e^{4t} & -\frac{2}{3}e^t + \frac{2}{3}e^{4t} \\ -\frac{1}{3}e^t + \frac{1}{3}e^{4t} & \frac{2}{3}e^t + \frac{1}{3}e^{4t} \end{pmatrix}$$

Method 2: eigen method.

Compute eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ as in Method 1. To get eigenvector $\mathbf{v}_1$,

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$3v_1 + 2v_2 = v_1$, $2v_2 = -2v_1$. Take $\mathbf{v}_1 = (1; -1)$.
To get eigenvector $\mathbf{v}_2$,

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$3v_1 + 2v_2 = 4v_1$, $2v_2 = v_1$. Take $\mathbf{v}_1 = (2; 1)$.
Therefore

$$A = VDV^{-1}, \quad V = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad V^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{4t} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^t + \frac{2}{3}e^{4t} & -\frac{2}{3}e^t + \frac{2}{3}e^{4t} \\ -\frac{1}{3}e^t + \frac{1}{3}e^{4t} & \frac{2}{3}e^t + \frac{1}{3}e^{4t} \end{pmatrix}$$

Method 3: Laplace transform.

$$sI - A = \begin{pmatrix} s-3 & -2 \\ -1 & s-2 \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s-1)(s-4)} \begin{pmatrix} s-2 & 2 \\ 1 & s-3 \end{pmatrix} = \begin{pmatrix} \frac{s-2}{(s-1)(s-4)} & \frac{2}{(s-1)(s-4)} \\ \frac{1}{(s-1)(s-4)} & \frac{s-3}{(s-1)(s-4)} \end{pmatrix}$$

Then do inverse Laplace transform for each element (by partial fractions) to get the answer.

(2)

$$\mathbf{x}(t) = e^{tA}\mathbf{x}(0) = \begin{pmatrix} \frac{1}{3}e^t + \frac{2}{3}e^{4t} & -\frac{2}{3}e^t + \frac{2}{3}e^{4t} \\ -\frac{1}{3}e^t + \frac{1}{3}e^{4t} & \frac{2}{3}e^t + \frac{1}{3}e^{4t} \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3}e^t - \frac{4}{3}e^{4t} \\ \frac{5}{3}e^t - \frac{2}{3}e^{4t} \end{pmatrix}$$

Problem 5. (15 points) Find e^{tA} for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This method cannot be solved by eigen method because A is not diagonalizable! (in fact, a generalization of eigen method works for this, but it is not included in the lecture. See the last part of section 3.5 of the textbook.)

Method 1: natural fundamental set method.

$$\det(zI - A) = \det \begin{pmatrix} z & -1 \\ 0 & z \end{pmatrix} = z^2$$

The roots are $r_1 = 0$ with multiplicity 2. The general solution to $y'' = 0$ is

$$y(t) = C_1 e^{0t} + C_2 t e^{0t} = C_1 + C_2 t, \quad y'(t) = C_2$$

For N_0,

$$1 = C_1, \quad 0 = C_2$$

$$N_0(t) = 1$$

For N_1,

$$0 = C_1, \quad 1 = C_2$$

$$N_1(t) = t$$

Therefore

$$e^{tA} = N_0(t)I + N_1(t)(A) = 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Method 2: Laplace transform.

$$sI - A = \begin{pmatrix} s & -1 \\ 0 & s \end{pmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2}\begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix}$$

Then do inverse Laplace transform for each element to get the answer.