Each group only needs to submit ONE file containing your solutions!

Problem 1. Solve the initial value problem by Laplace transform:

$$y'' + 2y' + 5y = 1, \quad y(0) = 1,\ y'(0) = 0$$

First, we have the formula

$$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}, \quad \mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$$

$$\begin{aligned}
s^2\mathcal{L}[y] - sy(0) - y'(0) + 2(s\mathcal{L}[y] - y(0)) + 5\mathcal{L}[y] &= \frac{1}{s} \\
(s^2+2s+5)\mathcal{L}[y] &= \frac{1}{s} + s + 2 = \frac{s^2+2s+1}{s} \\
\mathcal{L}[y] &= \frac{s^2+2s+1}{s(s^2+2s+5)} = \frac{C_1}{s} + \frac{C_2 s + C_3}{s^2+2s+5} \\
s^2+2s+1 &= C_1(s^2+2s+5) + s(C_2 s + C_3)
\end{aligned}$$

Compare $s^2, s, 1$ coefficients,

$$1 = C_1 + C_2,\ 2 = 2C_1 + C_3,\ 1 = 5C_1$$

$$C_1 = \frac{1}{5}, \quad C_2 = \frac{4}{5}, \quad C_3 = \frac{8}{5}$$

Therefore

$$\mathcal{L}[y] = \frac{s^2+2s+1}{s(s^2+2s+5)} = \frac{1}{5}\frac{1}{s} + \frac{\frac{4}{5}s + \frac{8}{5}}{s^2+2s+5} = \frac{1}{5}\frac{1}{s} + \frac{\frac{4}{5}(s+1) + \frac{4}{5}}{(s+1)^2+2^2}$$

Then

$$y(t) = \frac{1}{5} + \frac{4}{5}e^{-t}\cos(2t) + \frac{2}{5}e^{-t}\sin(2t)$$

Problem 2. Compute e^{tA} for

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

and express your result with real-valued functions.

(1) using the natural fundamental set method.

(2) using the eigen method.

(3) Use your result of e^{tA} to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

(1) The characteristic polynomial is

$$\det(zI - A) = \det \begin{pmatrix} z-1 & 2 \\ -1 & z-3 \end{pmatrix} = (z-1)(z-3)+2 = z^2-4z+5 = (z-(2+i))(z-(2-i))$$

Therefore the general solution to the related ODE $y'' - 4y' + 5y = 0$ is

$$y(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$$

and then

$$y'(t) = (2C_1 + C_2)e^{2t}\cos t + (2C_2 - C_1)e^{2t}\sin t$$

To get $N_0(t)$ which satisfies $N_0(0) = 1$, $N_0'(0) = 0$,

$$1 = C_1,\ 0 = 2C_1 + C_2 \quad \Rightarrow \quad C_1 = 1,\ C_2 = -2$$

$$N_0 = e^{2t}\cos t - 2e^{2t}\sin t$$

To get $N_1(t)$ which satisfies $N_1(0) = 0$, $N_1'(0) = 1$,

$$0 = C_1,\ 1 = 2C_1 + C_2 \quad \Rightarrow \quad C_1 = 0,\ C_2 = 1$$

$$N_1 = e^{2t}\sin t$$

Therefore

$$e^{tA} = (e^{2t}\cos t - 2e^{2t}\sin t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{2t}\sin t\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} e^{2t}\cos t - e^{2t}\sin t & -2e^{2t}\sin t \\ e^{2t}\sin t & e^{2t}\cos t + e^{2t}\sin t \end{pmatrix}$$

(2) Compute the eigenvalues $\lambda_1 = 2+i$, $\lambda_2 = 2-i$ as in (1). To find an eigenvector associated to $\lambda_1 = 2+i$,

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (2+i)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The first component gives $v_1 - 2v_2 = (2+i)v_1$, $-2v_2 = (1+i)v_1$. Therefore an eigenpair is

$$(2+i, \begin{pmatrix} -2 \\ 1+i \end{pmatrix})$$

Similarly, another eigenpair is

$$(2-i, \begin{pmatrix} -2 \\ 1-i \end{pmatrix})$$

Therefore $A = VDV^{-1}$ with

$$V = \begin{pmatrix} -2 & -2 \\ 1+i & 1-i \end{pmatrix}, \quad D = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$$

and (since $\det V = -2(1-i) + 2(1+i) = 4i$)

$$V^{-1} = \frac{1}{4i}\begin{pmatrix} 1-i & 2 \\ -1-i & -2 \end{pmatrix}$$

Therefore

$$
\begin{aligned}
e^{tA} =& V e^{tD} V^{-1} = \begin{pmatrix} -2 & -2 \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} e^{(2+i)t} & 0 \\ 0 & e^{(2-i)t} \end{pmatrix} \frac{1}{4i} \begin{pmatrix} 1-i & 2 \\ -1-i & -2 \end{pmatrix} \\
=& \frac{1}{4i} \begin{pmatrix} -2e^{(2+i)t} & -2e^{(2-i)t} \\ (1+i)e^{(2+i)t} & (1-i)e^{(2-i)t} \end{pmatrix} \begin{pmatrix} 1-i & 2 \\ -1-i & -2 \end{pmatrix} \\
=& e^{2t} \frac{1}{4i} \begin{pmatrix} -2e^{it} & -2e^{-it} \\ (1+i)e^{it} & (1-i)e^{-it} \end{pmatrix} \begin{pmatrix} 1-i & 2 \\ -1-i & -2 \end{pmatrix} \\
=& e^{2t} \frac{1}{4i} \begin{pmatrix} -2e^{it}(1-i) - 2e^{-it}(-1-i) & -4e^{it} + 4e^{-it} \\ (1+i)(1-i)e^{it} + (1-i)(-1-i)e^{-it} & 2(1+i)e^{it} - 2(1-i)e^{-it} \end{pmatrix} \\
=& e^{2t} \frac{1}{4i} \begin{pmatrix} 2i\cos t - 2i\sin t + 2i\cos t - 2i\sin t & -4i\sin t - 4i\sin t \\ 2i\sin t + 2i\sin t & 2i\cos t + 2i\sin t + 2i\sin t + 2i\cos t \end{pmatrix} \\
=& e^{2t} \begin{pmatrix} \cos t - \sin t & -2\sin t \\ \sin t & \cos t + \sin t \end{pmatrix}
\end{aligned}
$$

where in the second last equality we only keep the imaginary part in the matrix because the final result should be real.

$$
\mathbf{x}(t) = e^{tA}\mathbf{x}(0) = e^{2t} \begin{pmatrix} \cos t - \sin t & -2\sin t \\ \sin t & \cos t + \sin t \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = e^{2t} \begin{pmatrix} 3\cos t + \sin t \\ \sin t - 2\cos t \end{pmatrix}
$$