## Total time: 75 minutes

**Notice:**
**(1) Write your solution to each problem on a DIFFERENT answer sheet.**
**(2) Write your name on every answer sheet.**
**(3) You are allowed to use calculators, the textbook, and your notes in this exam.**
**(4) You are NOT allowed to use MATLAB or the internet in this exam.**
**(5) You do not need to write the interval of definition unless you are asked to do so.**

**Problem 1. (10 points)** Compute the interval of definition of the following initial value problem

$$t^2y'' + \frac{t}{t+2}y' + 3y + \ln(3-t) = 0, \quad y(1) = 1,\, y'(1) = -2$$

First rewrite into standard form

$$y'' + \frac{1}{t(t+2)}y' + \frac{3}{t^2}y = -\frac{\ln(3-t)}{t^2}, \quad y(1) = 1,\, y'(1) = -2$$

Bad points are 0, -2, and also $t < 3$ from the logarithm. Therefore the largest interval containing $t_I = 1$ is $(0, 3)$.

**Problem 2. (20 points)** Determine whether

$$Y_1(t) = t^2 + 2t, \quad Y_2(t) = 2t^2 - 3t + 1, \quad Y_3(t) = -t^2 + 12t - 2$$

are a fundamental set of solutions to the ODE

$$y''' = 0$$

<u>Justify your answer</u>.

(You are not required to prove that $Y_1, Y_2, Y_3$ are solutions.) To check whether they are a fundamental set, we need to check whether the Wronskian is zero. We could do that at any fixed $t$, for example, $t = 0$.

$$\begin{aligned} Wr[Y_1,Y_2,Y_3](0) &= \det\begin{pmatrix} t^2+2t & 2t^2-3t+1 & -t^2+12t-2 \\ 2t+2 & 4t-3 & -2t+12 \\ 2 & 4 & -2 \end{pmatrix}\Big|_{t=0} \\ &= \det\begin{pmatrix} 0 & 1 & -2 \\ 2 & -3 & 12 \\ 2 & 4 & -2 \end{pmatrix} = 0 \end{aligned}$$

Therefore $Y_1, Y_2, Y_3$ is not a fundamental set.
(Alternatively, you can also show this by noticing that $-3Y_1 + 2Y_2 + Y_3 = 0$.)

**Problem 3. (30=20+10 points)**

(1) Find the Green's function $g$ associated with the operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 4\frac{\mathrm{d}}{\mathrm{d}t} + 13$$

(in other words, $L$ is defined by $Ly = y'' + 4y' + 13y$.) Express your answer using real-valued functions.

(2) Use $g$ to compute a particular solution to

$$y'' + 4y' + 13y = e^{-2t}$$

(1) The Green's function $g$ is defined by

$$Lg = 0, \quad g(0) = 0,\ g'(0) = 1$$

The characteristic polynomial $p(z) = z^2 + 4z + 13$ has two simple roots (multiplicity is 1) $r_{1,2} = -2 \pm 3i$. Therefore the general solution to $Lg = 0$ is

$$g(t) = C_1 e^{-2t} \cos 3t + C_2 e^{-2t} \sin 3t$$

Then

$$g'(t) = (-2C_1 + 3C_2)e^{-2t} \cos 3t + (-3C_1 - 2C_2)e^{-2t} \sin 3t$$

Matching initial condition gives

$$0 = C_1, \quad 1 = -2C_1 + 3C_2 \quad \Rightarrow C_2 = \frac{1}{3}$$

Therefore

$$g(t) = \frac{1}{3} e^{-2t} \sin 3t$$

(2) One particular solution is given by (taking $t_I = 0$)

$$\begin{aligned} Y_P(t) &= \int_0^t g(t-s) f(s)\,\mathrm{d}s = \int_0^t \frac{1}{3} e^{-2(t-s)} \sin 3(t-s) \cdot e^{2s}\,\mathrm{d}s \\ &= \frac{1}{3} e^{-2t} \int \sin 3(t-s)\,\mathrm{d}s = \frac{1}{3} e^{-2t} \cdot \frac{1}{3} \cos 3(t-s)\big|_{s=0}^{t} \\ &= \frac{1}{9} e^{-2t} (1 - \cos 3t) \end{aligned}$$

**More problems on the next page!**

**Problem 4. (20 points)** Find the general solution to the ODE

$$y'' + 3y' - 4y + e^t(e^t - 1) = 0$$

Express your answer using <u>real-valued functions</u>.

First rewrite it into standard form

$$Ly = y'' + 3y' - 4y = -e^{2t} + e^t$$

The characteristic polynomial is $p(z) = z^2 + 3z - 4 = (z+4)(z-1)$ has two simple roots $r_1 = -4, r_2 = 1$.
For the RHS term $-e^{2t}$, 2 is not a root of $p(z)$. Therefore we do

$$L(e^{2t}) = p(2)e^{2t} = 6e^{2t}$$

$$L(-\frac{1}{6}e^{2t}) = p(2)e^{2t} = -e^{2t}$$

For the RHS term $e^{1\cdot t}$, 1 is a root of $p(z)$ with multiplicity 1. Therefore we do

$$L(e^t) = p(1)e^t = 0$$

$$L(te^t) = p'(1)e^t + p(1)te^t = 5e^t$$

$$L(\frac{1}{5}te^t) = e^t$$

Therefore a particular solution $Y_P(t) = -\frac{1}{6}e^{2t} + \frac{1}{5}te^t$. The general solution is

$$y(t) = -\frac{1}{6}e^{2t} + \frac{1}{5}te^t + C_1e^{-4t} + C_2e^t$$

(Alternatively, one can also solve this problem by using Green's function.)

**Problem 5. (20 points)** Find the general solution to the ODE

$$L(t)y = y'' - \frac{2t+2}{t^2+2t+2}y' + \frac{2}{t^2+2t+2}y = t^2 + 2t + 2$$

given that a fundamental set of solutions to $L(t)y = 0$ is

$$Y_1(t) = t + 1, \quad Y_2(t) = t^2 + 2t$$

We use the variation of parameters. Write

$$y(t) = u_1(t)Y_1(t) + u_2(t)Y_2(t)$$

Then $u_1, u_2$ satisfies

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

that is,

$$\begin{pmatrix} t+1 & t^2+2t \\ 1 & 2t+2 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t^2+2t+2 \end{pmatrix}$$

Solve this linear system and get

$$u_1' = -t^2 - 2t, \quad u_2' = t+1$$

Then integrate,

$$u_1 = \int (-t^2 - 2t)\,\mathrm{d}t = -\frac{1}{3}t^3 - t^2, \quad u_2 = \int (t+1)\,\mathrm{d}t = \frac{1}{2}t^2 + t$$

(omitting any $C$ here). Then a particular solution is

$$Y_P(t) = (-\frac{1}{3}t^3 - t^2)(t+1) + (\frac{1}{2}t^2 + t)(t^2+2t)$$

The general solution is

$$y(t) = (-\frac{1}{3}t^3 - t^2)(t+1) + (\frac{1}{2}t^2 + t)(t^2+2t) + C_1(t+1) + C_2(t^2+2t)$$