MATH 246 — Midterm Exam 1 Solution

29 Sept 2020, Instructor: Ruiwen Shu

Total time: 75 minutes

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators, the textbook, and your notes in this exam.
(4) You are NOT allowed to use MATLAB or the internet in this exam.
(5) You do not need to write the interval of definition unless you are asked to do so.

Problem 1. (15=10+5 points)

(1) (10 points) Identify all the linear differential equation(s) among the following, and explain why the others are not:

$$\text{(a): } \frac{\mathrm{d}x}{\mathrm{d}t} = t\sin x + 1, \quad \text{(b): } y'' = ty' + 3y, \quad \text{(c): } y' = y^2 + t^2, \quad \text{(d): } \partial_t u = \partial_{xx} u$$

(2) (5 points) Compute the interval of definition of the solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{t^2-1} + \frac{t}{t+2}, \quad y(0.1) = 1$$

(1) (b)(d) are linear. (a) is nonlinear because of the term $t\sin x$ (here x is the unknown function). (c) is nonlinear because of the term y^2.
(2) This is a first order linear ODE. The interval of definition is the largest interval containing $t_I = 0.1$ such that the coefficients $\frac{1}{t^2-1}$ and $\frac{t}{t+2}$ are continuous. The bad points are $t = \pm 1$, $t = -2$, and therefore the interval of definition is $(-1, 1)$.

Problem 2. (20=10+10 points)

(1) (10 points) Solve the initial value problem

$$t^2 yy' = y^2 + 1, \quad y(2) = 1$$

Express your solution in an explicit form.

(2) (10 points) Use the explicit Euler method to approximate $y(2.6)$ of the above initial value problem, with step size $h = 0.2$. Keep at least 3 digits after the decimal point, in all your intermediate results.

(1) This is a separable equation.

$$\frac{y\,\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}t}{t^2}$$

$$\int \frac{y\,\mathrm{d}y}{y^2+1} = \int \frac{\mathrm{d}t}{t^2}$$
$$\frac{1}{2}\ln(y^2+1) + C = -\frac{1}{t}$$
where left hand side uses a substitution $u = y^2 + 1$. To determine C, use initial condition $t = 2, y = 1$ to get
$$\frac{1}{2}\ln(1^2+1) + C = -\frac{1}{2} \Rightarrow C = -\frac{1}{2} - \frac{1}{2}\ln 2$$
Therefore
$$\frac{1}{2}\ln(y^2+1) - \frac{1}{2} - \frac{1}{2}\ln 2 = -\frac{1}{t}$$
Then solve for y:
$$\frac{1}{2}\ln(y^2+1) = \frac{1}{2} + \frac{1}{2}\ln 2 - \frac{1}{t}$$
$$\ln(y^2+1) = 1 + \ln 2 - \frac{2}{t}$$
$$y^2 + 1 = e^{1+\ln 2 - \frac{2}{t}}$$
$$y^2 = e^{1+\ln 2 - \frac{2}{t}} - 1$$
$$y = \sqrt{e^{1+\ln 2 - \frac{2}{t}} - 1}$$
The last square root is taken positive because the initial condition $y(2) = 1 > 0$.
(2) $t_0 = t_I = 2$, $t_1 = 2.2$, $t_2 = 2.4$, $t_3 = 2.6 = t_F$. Rewrite the ODE as
$$y' = \frac{y^2+1}{t^2 y} =: f(t, y)$$
$$y_0 = 1$$
$$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.2\frac{1^2+1}{2^2 \cdot 1} = 1.1$$
$$y_2 = y_1 + hf(t_1, y_1) = 1.1 + 0.2\frac{1.1^2+1}{2.2^2 \cdot 1.1} \approx 1.183$$
$$y_3 = y_2 + hf(t_2, y_2) = 1.183 + 0.2\frac{1.183^2+1}{2.4^2 \cdot 1.183} \approx 1.253$$
The approximation to $y(2.6)$ is 1.253. (the final answer might be slightly different if you keep more digits in intermediate results.)

Problem 3. (20 points) Find the general solution to the second order ODE
$$x'' + 2x' + t = 0$$

Let $u = x'$. Then u satisfies
$$u' + 2u = -t$$

which is linear.

$$a(t) = 2, \quad A(t) = \int 2\,\mathrm{d}t = 2t, \quad \text{integrating factor: } e^{2t}$$

$$(e^{2t}u)' = -te^{2t}$$

$$e^{2t}u = -\int te^{2t}\,\mathrm{d}t = -\frac{1}{2}te^{2t} + \frac{1}{2}\int e^{2t}\,\mathrm{d}t = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + C_1$$

using integration by parts.

$$u = -\frac{1}{2}t + \frac{1}{4} + C_1e^{-2t}$$

Since $x' = u$, we integrate and get

$$x = \int (-\frac{1}{2}t + \frac{1}{4} + C_1e^{-2t})\,\mathrm{d}t = -\frac{1}{4}t^2 + \frac{1}{4}t - \frac{C_1}{2}e^{-2t} + C_2$$

Problem 4. (20 points) Find the general solution to the ODE

$$(2y + x^5)\,\mathrm{d}x + (y^3 + 2x)\,\mathrm{d}y = 0$$

You may leave your final answer as an implicit function.

$M = 2y + x^5$, $N = y^3 + 2x$.

$$\partial_y M = 2, \quad \partial_x N = 2 = \partial_y M$$

Therefore the differential form is exact. By $\partial_x H = M$ and viewing y as a constant,

$$H = \int (2y + x^5)\,\mathrm{d}x = 2xy + \frac{1}{6}x^6 + h(y)$$

Taking y derivative and using $\partial_y H = N$,

$$2x + h'(y) = y^3 + 2x, \quad h'(y) = y^3, \quad h(y) = \frac{1}{4}y^4$$

Therefore $H = 2xy + \frac{1}{6}x^6 + \frac{1}{4}y^4$. The general solution is

$$2xy + \frac{1}{6}x^6 + \frac{1}{4}y^4 = C$$

Problem 5. (25=15+5+5 points) Initially a tank contains 10 liters of water. At some instant, salt solution of concentration 2g/L starts to flow into the tank at a rate of 3L/h, while well-stirred mixture is flowing out, so that the volume of liquid in the tank stays the same.

(1) (15 points) Compute the concentration of salt in the tank $C(t)$.

(2) (5 points) Will the concentration of salt in the tank reach 1g/L? If yes, compute the first time this happens. If no, explain your answer.

(3) (5 points) Will the concentration of salt in the tank reach 3g/L? If yes, compute the first time this happens. If no, explain your answer.

(1) The total volume $V(t) = 10$ is constant.
The inflow has rate 3 with concentration 2. The outflow has rate 3 (to keep the total volume the same) and concentration $C(t)$. Therefore

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 3 \cdot 2 - 3C(t) = 6 - 3\frac{S(t)}{V(t)} = 6 - 0.3S$$

This is a linear equation in $S(t)$.

$$\frac{\mathrm{d}S}{\mathrm{d}t} + 0.3S = 6$$

$$a(t) = 0.3, \quad A(t) = \int 0.3\,\mathrm{d}t = 0.3t, \quad \text{integrating factor: } e^{0.3t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{0.3t}S) = 6e^{0.3t}$$

$$e^{0.3t}S = \int 6e^{0.3t}\,\mathrm{d}t = \frac{6}{0.3}e^{0.3t} + C_1 = 20e^{0.3t} + C_1$$

(don't use C here to avoid confusion with $C(t)$)

$$S = 20 + C_1e^{-0.3t}$$

Initially there is no salt in the tank, so initial condition is $S(0) = 0$. Therefore

$$0 = 20 + C_1e^{-0.3 \cdot 0} \Rightarrow C_1 = -20$$

Therefore

$$S(t) = 20 - 20e^{-0.3t}$$

$$C(t) = \frac{20 - 20e^{-0.3t}}{10} = 2 - 2e^{-0.3t}$$

(2) Let $C(t) = 2 - 2e^{-0.3t} = 1$ and solve for t:

$$2e^{-0.3t} = 1$$

$$e^{-0.3t} = \frac{1}{2}$$

$$-0.3t = \ln\frac{1}{2}$$

$$t = \frac{\ln\frac{1}{2}}{-0.3}$$

(notice this answer is equal to $\frac{10\ln 2}{3} \approx 2.310$)
(3) Let $C(t) = 2 - 2e^{-0.3t} = 3$ and solve for t:

$$2e^{-0.3t} = -1$$

This can never happen because the left hand side is always positive. Therefore the concentration of salt cannot reach 3.
(You may also explain as: since the initial concentration is 0 and the inflow concentration is 2, the concentration in the tank can never exceed 2, and therefore cannot reach 3.)