Each group only needs to submit ONE file containing your solutions!

Problem 1. Find the general solution to the ODE

$$y''' - y = t\cos t$$

by using Key Identity Evaluation. (you may need to express $\cos t$ in terms of complex exponentials.)
The characteristic polynomial is $p(z) = z^3 - 1$ which has simple roots $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$. Therefore the general solution to the homogeneous equation is

$$y(t) = C_1 e^t + C_2 e^{\frac{-1+\sqrt{3}i}{2}t} + C_3 e^{\frac{-1+\sqrt{3}i}{2}t} = C_1 e^t + \tilde{C}_2 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \tilde{C}_3 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

(if you want a real form.)
Write

$$t\cos t = \frac{1}{2}te^{it} + \frac{1}{2}te^{-it}$$

For the term $\frac{1}{2}te^{it}$, $\mu = i$ is not a root of $p(z)$, and $d = 1$. Therefore we use the identities

$$L(e^{it}) = p(i)e^{it} = (-i-1)e^{it}$$

$$L(te^{it}) = p'(i)e^{it} + p(i)te^{it} = -3e^{it} + (-i-1)te^{it}$$

Therefore

$$L(\frac{1}{-i-1}\cdot\frac{1}{2}te^{it}) = \frac{1}{-i-1}\cdot\frac{1}{2}\cdot(-3e^{it}) + \frac{1}{2}te^{it}$$

$$L\Big(\frac{1}{-i-1}\cdot\frac{1}{2}te^{it} - \frac{1}{-i-1}\cdot\frac{1}{-i-1}\cdot\frac{1}{2}\cdot(-3)e^{it}\Big) = \frac{1}{2}te^{it}$$

Therefore a particular solution to $Ly = \frac{1}{2}te^{it}$ is

$$Y_{P,1}(t) = \frac{1}{-i-1}\cdot\frac{1}{2}te^{it} - \frac{1}{-i-1}\cdot\frac{1}{-i-1}\cdot\frac{1}{2}\cdot(-3)e^{it} = \frac{i-1}{4}te^{it} - \frac{3}{4}ie^{it}$$

(in exam, such simplification is not required)
Similarly, a particular solution to $Ly = \frac{1}{2}te^{-it}$ is

$$Y_{P,2}(t) = \frac{-i-1}{4}te^{-it} + \frac{3}{4}ie^{-it}$$

Therefore, the general solution is

$$y(t) = \frac{i-1}{4}te^{it} - \frac{3}{4}ie^{it} + \frac{-i-1}{4}te^{-it} + \frac{3}{4}ie^{-it} + C_1e^t + \tilde{C}_2 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + \tilde{C}_3 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

(you could further write it into real form.)

Problem 2. Consider the ODE

$$Ly = y'' - y = \frac{1}{1+e^{2t}}$$

(1) Find the Green's function associated with L.

(2) Use the Green's function to give a particular solution.

(1) The Green's function g is the solution to $Lg = 0$, $g(0) = 0$, $g'(0) = 1$. $p(z) = z^2 - 1 = (z+1)(z-1)$. Therefore the general solution to $Lg = 0$ is

$$g(t) = C_1 e^{-t} + C_2 e^t$$

Matching the initial condition gives

$$0 = C_1 + C_2, \quad 1 = -C_1 + C_2$$

Solve and get

$$C_1 = -\frac{1}{2}, \quad C_2 = \frac{1}{2}$$

Therefore

$$g(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^t$$

(2) A particular solution is (taking $t_I = 0$)

$$\begin{aligned} Y_P(t) &= \int_0^t g(t-s)f(s)\,\mathrm{d}s = -\frac{1}{2}\int_0^t e^{s-t}\frac{1}{1+e^{2s}}\,\mathrm{d}s + \frac{1}{2}\int_0^t e^{t-s}\frac{1}{1+e^{2s}}\,\mathrm{d}s \\ &= -\frac{1}{2}e^{-t}\int_0^t \frac{e^s}{1+e^{2s}}\,\mathrm{d}s + \frac{1}{2}e^t\int_0^t \frac{e^{-s}}{1+e^{2s}}\,\mathrm{d}s \end{aligned}$$

The two integrals above are (omitting $+C$, using $u = e^s$)

$$\int \frac{e^s}{1+e^{2s}}\,\mathrm{d}s = \int \frac{1}{1+u^2}\,\mathrm{d}u = \tan^{-1} u = \tan^{-1}(e^s)$$

$$\int \frac{e^{-s}}{1+e^{2s}}\,\mathrm{d}s = \int \frac{1}{u^2(1+u^2)}\,\mathrm{d}u = \int (\frac{1}{u^2} - \frac{1}{1+u^2})\,\mathrm{d}u = -\frac{1}{u} - \tan^{-1} u = -e^{-s} - \tan^{-1}(e^s)$$

Therefore

$$\begin{aligned} Y_P(t) &= -\frac{1}{2}e^{-t}\tan^{-1}(e^s)_{s=0}^t + \frac{1}{2}e^t(-e^{-s} - \tan^{-1}(e^s))_{s=0}^t \\ &= -\frac{1}{2}e^{-t}(\tan^{-1}(e^t) - \frac{\pi}{4}) + \frac{1}{2}e^t(-e^{-t} - \tan^{-1}(e^t) + 1 + \frac{\pi}{4}) \end{aligned}$$