Each group only needs to submit ONE file containing your solutions!

Problem 1. A tank of capacity 10L is full with water. At some instant, sugar solution with concentration 3g/L flows into the tank at a rate of 0.5L/h. At the same time, well-stirred mixture flows out of the tank at a rate of 1.5L/h. What is the concentration of sugar when the tank is half full?

Let $V(t)$ be the volume of liquid in the tank, $S(t)$ be the total mass of sugar, and $C(t) = S(t)/V(t)$ be the concentration.
$V(t)$ satisfies the ODE

$$\frac{dV}{dt} = 0.5 - 1.5 = 1, \quad V(0) = 10$$

which gives

$$V(t) = 10 - t$$

$S(t)$ satisfies the ODE

$$\frac{dS}{dt} = 3 \cdot 0.5 - C(t) \cdot 1.5 = 1.5 - 1.5 \cdot \frac{S}{10-t}, \quad S(0) = 0$$

($S(0) = 0$ because initially there is no sugar in the tank)
This is a linear ODE. Put it into standard form

$$\frac{dS}{dt} + \frac{1.5}{10-t} S = 1.5$$

$$a(t) = \frac{1.5}{10-t}, \quad A(t) = \int \frac{1.5}{10-t} dt = -1.5 \ln(10-t)$$

(we don't need $\ln|10-t|$ because $10-t$ is the volume, so non-negative.) Therefore the integrating factor is $e^{-1.5\ln(10-t)} = (10-t)^{-1.5}$.

$$\frac{d}{dt}((10-t)^{-1.5} S) = 1.5(10-t)^{-1.5}$$

$$(10-t)^{-1.5} S = \int 1.5(10-t)^{-1.5} dt = 3(10-t)^{-0.5} + C$$

$$S = 3(10-t) + C(10-t)^{1.5}$$

Use the initial condition $S(0) = 0$,

$$0 = 3(10-0) + C(10-0)^{1.5} \Rightarrow C = -\frac{30}{10^{1.5}}$$

Therefore

$$S = 3(10-t) - \frac{30}{10^{1.5}}(10-t)^{1.5}$$

$$C(t) = S(t)/V(t) = 3 - \frac{30}{10^{1.5}}(10-t)^{0.5}$$

It is asked 'when the tank is half full', which means $V(t) = 10 - t = 10/2 = 5$, $t = 5$. Therefore the answer is

$$C(5) = 3 - \frac{30}{10^{1.5}} 5^{0.5}$$

Problem 2. The population of cats $p_{cat}(t)$ grows exponentially with a growth rate $r_{cat} > 0$. The population of rats $p_{rat}(t)$ grows exponentially with a growth rate $r_{rat} > 0$, in the absence of cats. When cats and rats are together, the rats are subject to a harvest rate $h(t) = 5\sqrt{p_{cat}(t)}$.

(1) If the initial population of cats is 1000, compute $p_{cat}(t)$ (your answer should involve r_{cat}).

By exponential model formula,

$$p_{cat}(t) = 1000 e^{r_{cat} t}$$

(2) If cats and rats are together, and their initial population are 1000 and 3000 respectively, compute $p_{rat}(t)$ (your answer should involve r_{cat} and r_{rat}).

From now on, we omit any 'rat' subscript.
$p(t)$ satisfies the ODE

$$\frac{dp}{dt} = rp - 5\sqrt{p_{cat}} = rp - 5\sqrt{1000} e^{\frac{r_{cat}}{2} t}$$

This is a linear ODE, and its integrating factor is e^{-rt}.

$$\frac{d}{dt}(e^{-rt} p) = -5\sqrt{1000} e^{\frac{r_{cat}}{2} t} e^{-rt}$$

$$e^{-rt} p = -5\sqrt{1000} \int e^{(\frac{r_{cat}}{2} - r)t} dt$$

- If $\frac{r_{cat}}{2} - r \neq 0$, then

$$e^{-rt} p = \frac{-5\sqrt{1000}}{\frac{r_{cat}}{2} - r} e^{(\frac{r_{cat}}{2} - r)t} + C = A e^{(\frac{r_{cat}}{2} - r)t} + C, \quad A := \frac{-5\sqrt{1000}}{\frac{r_{cat}}{2} - r}$$

$$p = A e^{\frac{r_{cat}}{2} t} + C e^{rt}$$

  Use the initial condition $p(0) = 3000$,

$$3000 = A + C \Rightarrow C = 3000 - A$$

  Therefore

$$p = A e^{\frac{r_{cat}}{2} t} + (3000 - A) e^{rt}$$

- If $\frac{r_{cat}}{2} - r = 0$, then

$$e^{-rt} p = -5\sqrt{1000} t + C$$
$$p = (-5\sqrt{1000} t + C) e^{rt}$$

  Use the initial condition $p(0) = 3000$,

$$3000 = C$$

  Therefore

$$p = (-5\sqrt{1000} t + 3000) e^{rt}$$

(3) Under the same assumption as (2), in order to avoid rats from being extinct, what condition should r_{cat} and r_{rat} satisfy?

- If $\frac{r_{cat}}{2} - r > 0$, then $A < 0$. Inside

$$p = Ae^{\frac{r_{cat}}{2}t} + (3000 - A)e^{rt}$$

  the first term is a negative exponential growth, while the second term is a positive exponential growth. Since $\frac{r_{cat}}{2} > r$ in this case, the first term dominates when t is large, which means the rats will eventually be extinct.

- If $\frac{r_{cat}}{2} - r < 0$, then $A > 0$. Inside

$$p = Ae^{\frac{r_{cat}}{2}t} + (3000 - A)e^{rt}$$

  the first term is a positive exponential growth, while the second term is an exponential growth, whose sign depending on the sign of $3000 - A$. Since $\frac{r_{cat}}{2} < r$ in this case, the second term dominates when t is large. Therefore, when $A \leq 3000$, then $p(t)$ is always positive, no extinction; when $A > 3000$, $p(t)$ is dominated by a negative exponential growth, which gives extinction.

- If $\frac{r_{cat}}{2} - r < 0$, then

$$p = (-5\sqrt{1000}t + 3000)e^{rt}$$

  which becomes negative for large t, which gives extinction.

In conclusion, if we don't want extinction, then we need $\frac{r_{cat}}{2} - r < 0$ and $A \leq 3000$, which is

$$\frac{-5\sqrt{1000}}{\frac{r_{cat}}{2} - r} \leq 3000 \Leftrightarrow \frac{r_{cat}}{2} \leq r - \frac{5\sqrt{1000}}{3000}$$