8.1 Discrete least square approximation

Given (x_i, y_i), $i = 1, \cdots, m$,

$y_i \approx f(x_i)$, want a polynomial

to approximate $f(x)$.

- Lagrange interpolation: find $P(x)$ w/ $\deg \leq m-1$ s.t.

$$P(x_i) = y_i \quad , \quad i = 1, \cdots, m$$

Issues:

① When incoming data is noisy, the Lagrange interp. $P(x)$ can be oscillatory.

② Sometimes it's better to use low deg. poly. because coeffs can have physical meanings (linear, quadratic) please move this page to 2 pages later.

Question : Given (x_i, y_i), $i=1, \cdots, m$,

find $P(x) = \sum_{k=0}^{n} a_k x^k$ (w/ $n \le m-1$ given)

s.t. $E_2(a_0, \cdots, a_n)$ is minimized.

Case $n=1$: (linear least square)

$$P(x) = a_0 + a_1 x$$

$$E_2(a_0, a_1) = \sum_{i=1}^{m} \left(y_i - (a_0 + a_1 x_i) \right)^2$$

To minimize E_2, look for critical points.

Idea: find a lower deg. poly.

$$P(x) = \sum_{k=0}^{n} a_k x^k, \quad n \le m-1, \quad \text{s.t.}$$

$|y_i - P(x_i)|$, $i = 1, \cdots, m$ are as small as possible.

How to measure "smallness"?

$$E_\infty(a_0, \cdots, a_n) = \max_{1 \le i \le m} \left| y_i - \sum_{k=0}^{n} a_k x_i^k \right|$$

$$E_1(a_0, \cdots, a_n) = \sum_{i=1}^{m} \left| y_i - \sum_{k=0}^{n} a_k x_i^k \right|$$

(absolute deviation)

$$E_2(a_0, \cdots, a_n) = \sum_{i=1}^{m} \left(y_i - \sum_{k=0}^{n} a_k x_i^k \right)^2$$

(least square)

(easy to compute)

$$0 = \frac{\partial E_2}{\partial a_0} = \sum_{i=1}^{m} 2\left(y_i - (a_0 + a_1 x_i)\right) \cdot (-1)$$

$$0 = \frac{\partial E_2}{\partial a_1} = \sum_{i=1}^{m} 2\left(y_i - (a_0 + a_1 x_i)\right) \cdot (-x_i)$$

$$0 = \sum_{i=1}^{m} (y_i - a_0 - a_1 x_i)$$

$$0 = \sum_{i=1}^{m} (x_i y_i - x_i a_0 - x_i^2 a_1)$$

$$\begin{cases} m\,a_0 + \left(\sum_{i=1}^{m} x_i\right) a_1 = \sum_{i=1}^{m} y_i \\ \left(\sum_{i=1}^{m} x_i\right) a_0 + \left(\sum_{i=1}^{m} x_i^2\right) a_1 = \sum_{i=1}^{m} x_i y_i \end{cases}$$

$$\Rightarrow \quad a_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \cdot \sum x_i}{m \sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_1 = \frac{m \sum x_i y_i - \sum x_i \cdot \sum y_i}{m \sum x_i^2 - \left(\sum x_i\right)^2}$$

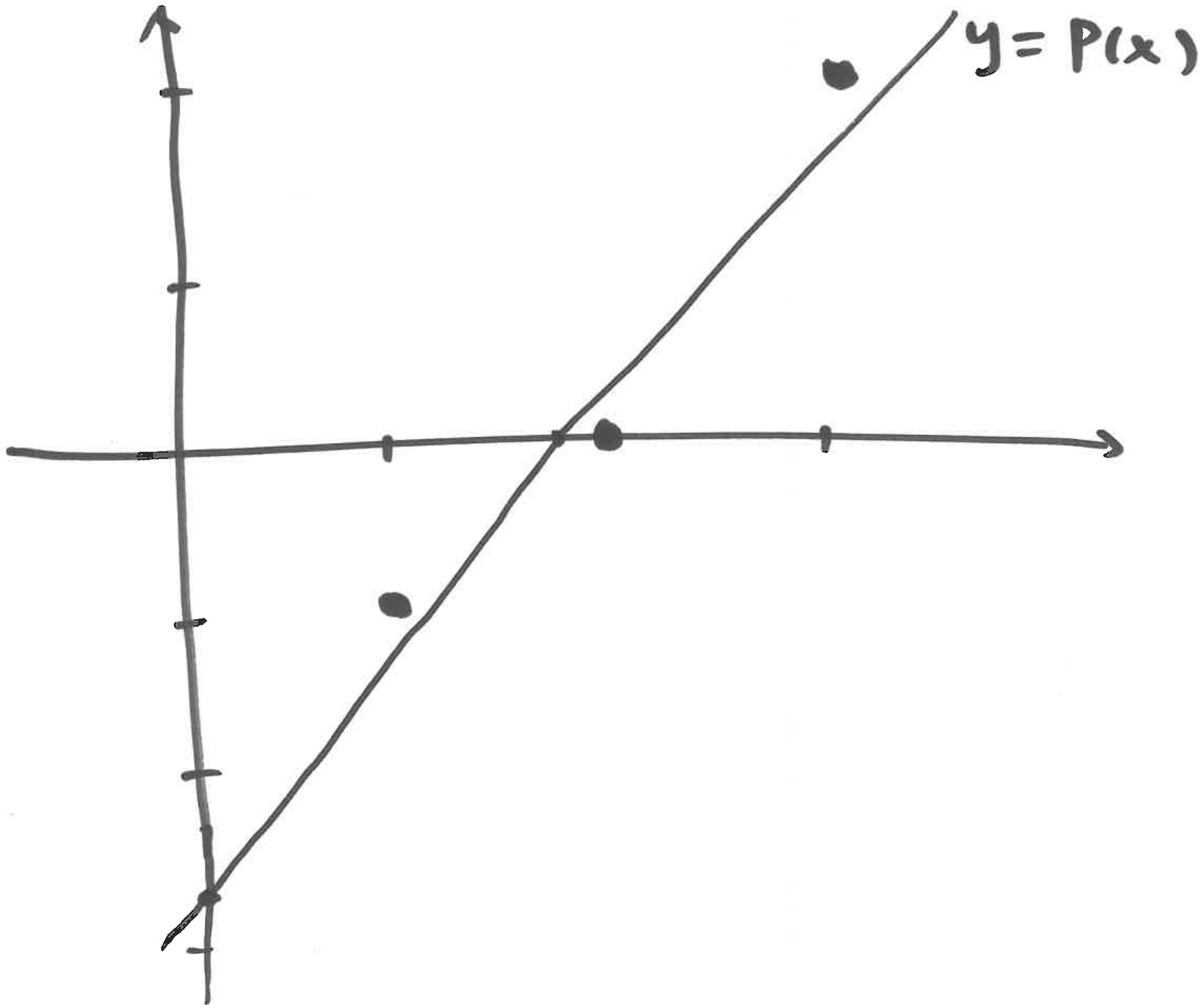
Ex Find linear least square of

$(1,-1)$, $(2,0)$, $(3,2)$

$$P(x) = a_0 + a_1 x$$

$$\begin{cases} 3a_0 + 6a_1 = 1 \\ 6a_0 + 14a_1 = 5 \end{cases} \rightsquigarrow \begin{cases} a_0 = -\frac{8}{3} \\ a_1 = \frac{3}{2} \end{cases}$$

$$P(x) = -\frac{8}{3} + \frac{3}{2}x$$


y = P(x)

General case:

$$E_2(a_0, \cdots, a_n) = \sum_{i=1}^{m} \left(y_i - \sum_{k=0}^{n} a_k x_i^k \right)^2$$

critical pts:

$$0 = \frac{\partial E_2}{\partial a_j} = \sum_{i=1}^{m} 2 \left(y_i - \sum_{k=0}^{n} a_k x_i^k \right) \cdot \left(- x_i^j \right)$$

$$j = 0, \cdots, n$$

$$0 = \sum_{i=1}^{m} \left(y_i x_i^j - \sum_{k=0}^{n} a_k x_i^{k+j} \right)$$

$$\sum_{k=0}^{n} \left(\sum_{i=1}^{m} x_i^{k+j} \right) \cdot a_k = \sum_{i=1}^{m} y_i x_i^j, \quad j = 0, \cdots, n$$

$$\begin{pmatrix} \sum x_i^0 & \sum x_i^1 & \cdots & \sum x_i^n \\ \sum x_i^1 & \sum x_i^2 & \cdots & \sum x_i^{n+1} \\ \vdots & \vdots & & \vdots \\ \sum x_i^n & \sum x_i^{n+1} & \cdots & \sum x_i^{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \sum y_i x_i^0 \\ \sum y_i x_i^1 \\ \vdots \\ \sum y_i x_i^n \end{pmatrix}$$

↳ this matrix is pos. def., thus invertible.