6.6 Some special matrices

- LU decomp. of symmetric matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}$$

$$a_{ij} = a_{ji} \qquad\qquad A^T = A$$

Thm If A is symmetric and invertible, and has LU decomp., then

$$U = DL^T \quad \text{w/} \quad D = \mathrm{diag}\{a_{11}^{(1)}, a_{22}^{(2)}, \dots, a_{nn}^{(n)}\}$$

- Then $A = LU = LDL^T$

"LDL^T decomposition"

Pf) We first show that every intermediate step of G. e. is symmetric.

It suffices to prove for step 1.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad a_{ij} = a_{ji}$$

Step 1: $E_j \rightsquigarrow E_j - m_{j1} E_1$

$$m_{j1} = \frac{a_{j1}}{a_{11}}$$

$j = 2, \cdots, n$

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}$$

want to show sym.

need to show: $a_{ij}^{(2)} = a_{ji}^{(2)}$ $\quad i,j \in \{2,\dots,n\}$

$$a_{ij}^{(2)} = a_{ij} - m_{i1} \cdot a_{1j} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$$

$$a_{ji}^{(2)} = a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} = a_{ij} - \frac{a_{1j}}{a_{11}} a_{i1}$$

sym. of A

To see $U = DL^T$,

$$DL^T = \begin{pmatrix} a_{11}^{(1)} & & & \\ & a_{22}^{(2)} & & \\ & & \ddots & \\ & & & a_{nn}^{(n)} \end{pmatrix} \begin{pmatrix} 1 & m_{21} & \cdots & m_{n1} \\ & 1 & \ddots & \vdots \\ & & \ddots & m_{n,n-1} \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}^{(1)} & a_{11}^{(1)} m_{21} & \cdots & a_{11}^{(1)} m_{n1} \\ & a_{22}^{(2)} & a_{22}^{(2)} m_{32} \cdots & a_{22}^{(2)} m_{n2} \\ & & \ddots & \vdots \\ & & & a_{n-1,n-1}^{(n-1)} m_{n,n-1} \\ & & & a_{nn}^{(n)} \end{pmatrix}$$

Recall:

$$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & \cdots & a_{1n}^{(1)} \\ & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} \\ & & \ddots & & \vdots \\ & & & & a_{nn}^{(n)} \end{pmatrix}$$

(i,j)-element of DL^T:

$$a_{ii}^{(i)} m_{ji} = a_{ii}^{(i)} \cdot \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}} = a_{ji}^{(i)}$$

(i,j)-element of U: $\quad a_{ij}^{(i)} \; \|$ $\Leftarrow$ sym. of intermediate steps.

Ex $A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 3 \\ -2 & 3 & 6 \end{pmatrix}$

Compute LDL^T decomp. of A.

Step 1: $m_{21} = -\frac{1}{2}$

$m_{31} = -1$

$$\begin{pmatrix} 2 & -1 & -2 \\ 0 & \frac{3}{2} & 2 \\ 0 & \boxed{2} & 4 \end{pmatrix}$$

don't need to compute, by sym.

Step 2: $m_{32} = \frac{4}{3}$

$$U = \begin{pmatrix} 2 & -1 & -2 \\ 0 & \frac{3}{2} & 2 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 \\ 0 & \frac{3}{2} & 2 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

$$A = LDL^T$$

$$= \begin{pmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ -1 & \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ & 1 & \frac{4}{3} \\ & & 1 \end{pmatrix}$$

- For symmetric matrix, by symmetry of intermediate steps, G.e. has computational cost only <u>half</u> of original cost.

Positive definite matrix

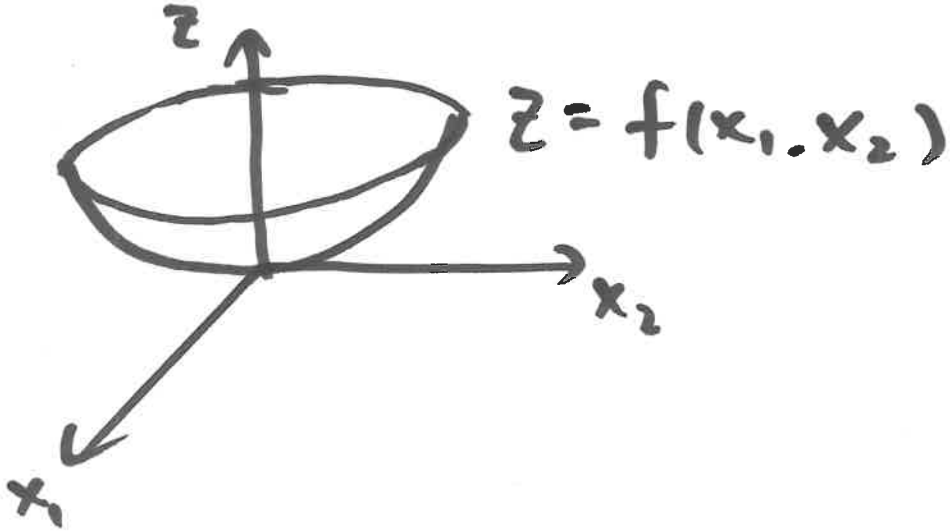
Def A matrix A is positive definite

if A is symmetric, and

$$x^T A x > 0 \quad , \quad \forall \; 0 \neq x \in \mathbb{R}^n$$

(in other words, $f(x) = x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j > 0$)

case $n=2$: $\quad f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$



For symmetric matrix A, the following are equivalent:

- A is pos. def.

- Every leading principal submatrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} \quad \text{has } \det > 0, \quad k = 1, \cdots, n$$

- All eigenvalues of A are positive

  ↳ $\lambda \in \mathbb{C}$ s.t. $Ax = \lambda x$ for some $x \neq 0$

  for sym. matrix, all eigenvalues are real.

Thm. Pos. def. matrix always has LDL^T decomp. w/ $a_{ii}^{(i)} > 0$, $i = 1, \cdots, n$

- Using this. $D = \mathrm{diag}\{a_{11}^{(1)}, \cdots, a_{nn}^{(n)}\}$

write $\tilde{D} = \mathrm{diag}\{\sqrt{a_{11}^{(1)}}, \cdots \sqrt{a_{nn}^{(n)}}\}$

then $\tilde{D}^2 = D$

$$A = LDL^T = L\tilde{D} \cdot \tilde{D}^T L^T = (L\tilde{D}) \cdot (L\tilde{D})^T$$
$$= \tilde{L} \cdot \tilde{L}^T$$

$\boxed{\tilde{L} := L\tilde{D} \text{ lower triangular}}$

Ex Check pos. def. of previous A

$$\det\begin{pmatrix} 2 \end{pmatrix} = 2 > 0$$

$$\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 2\cdot 2 - (-1)\cdot(-1) = 3 > 0$$

$$\det\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 3 \\ -2 & 3 & 6 \end{pmatrix} = 4 > 0$$

$\Rightarrow$ A is pos. def.