6.5 Matrix factorization

<u>Question</u>: To solve $Ax = b$, m times,

w/ the same A and different b.

- Direct computation by G. e.

  cost: $O(mn^3)$

  (but most cost are spent on A)

⇝ Idea: only do these operations on A once, keep record.

Notation: $a_{ij}^{(k)}$ denotes the (i,j) element of A before the k-th step of G.e.

Ignore "$a_{ii} = 0$" issue.

$$A^{(1)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix}$$

$$m_{j1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}$$

Step 1:

$E_j \rightsquigarrow E_j - m_{j1} E_1$

$j = 2, \cdots, n$

$$A^{(2)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}$$

Define $$M^{(1)} = \begin{pmatrix} 1 & & & & \\ -m_{21} & 1 & & & \\ -m_{31} & & 1 & & \\ \vdots & & & \ddots & \\ -m_{n1} & & & & 1 \end{pmatrix}$$

$$A^{(2)} = M^{(1)} A^{(1)}$$

Similarly, for i-th step $(i = 1, \cdots, n-1)$

$$M^{(i)} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & -m_{i+1,i} & 1 & & & \\ & & -m_{i+2,i} & & 1 & & \\ & & \vdots & & & \ddots & \\ & & -m_{ni} & & & & 1 \end{pmatrix} \quad \leftarrow i+1 \text{ row}$$

$\uparrow$ i column

$$A^{(i+1)} = M^{(i)} A^{(i)}$$

$$m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$$

$$j = i+1, \cdots, n$$

$M^{(i)}$: Gaussian tranformation matrix

Finally, upper triangular form:

$$U = A^{(n)} = M^{(n-1)} A^{(n-1)}$$

$$= M^{(n-1)} M^{(n-2)} A^{(n-2)}$$

$$= \cdots = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A$$

Denote $L^{(i)} = \left(M^{(i)}\right)^{-1}$

$$= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,i} & 1 & & \\ & & \vdots & & \ddots & \\ & & m_{ni} & & & 1 \end{pmatrix}$$

$$A = \underbrace{L^{(1)} L^{(2)} \cdots L^{(n-1)}}_{L} U = LU$$

$$L = \begin{pmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ m_{n1} & m_{n2} & \cdots & m_{n,n-1} & 1 \end{pmatrix} \qquad m_{ji} = \frac{a_{ji}^{(i)}}{a_{ii}^{(i)}}$$

<u>Thm</u> If G.e. can be applied to A without row exchange, then

$A = LU$, L lower triangular, U upper triangular.

"LU decomposition/factorization"

Given the LU decomposition $A = LU$,

to solve $Ax = b$,

$$L \underbrace{U x}_{y} = b$$

① Solve $Ly = b$

$$\begin{pmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{n2} & \cdots & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$y_1 = b_1$

$y_2 = b_2 - m_{21} y_1$

$\vdots$

$$y_i = b_i - \sum_{j=1}^{i-1} m_{ij} y_j$$

similar to backward substitution

② Solve $Ux = y$

backward substitution

Comp. cost of solving $Ax = b$, m times, same A, different b, using LU decomp.:

① Compute $A = LU$ cost: $O(n^3)$

② Use LU decomp. to solve $Ax = b$,

cost: $O(n^2)$ for each b

$\Rightarrow$ Total cost: $O(n^3 + mn^2)$

Ex $A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -1 & -4 & 1 \end{pmatrix}$ $b = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

Use LU decomp. to solve $Ax = b$.

Step 1: $m_{21} = 2$

$m_{31} = -1$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & -6 & 2 \end{pmatrix}$$

Step 2: $m_{32} = -2$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & -4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & -2 & 1 \end{pmatrix}$$

To solve $Ax = b$:

① $Ly = b$

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$y_1 = -1$$

$$y_2 = 2 - 2y_1 = 2 - 2\cdot(-1) = 4$$

$$y_3 = 3 + y_1 + 2y_2 = 3 + (-1) + 2\cdot 4 = 10$$

2 $Ux = y$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 10 \end{pmatrix}$$

same as last time, backward subs.

- Compute $\det(A)$:

$$A = LU$$

$$\det(A) = \det(L)\cdot\det(U) = a_{11}^{(1)} a_{22}^{(2)} \cdots a_{nn}^{(n)}$$

$$U = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ & & \ddots & \vdots \\ & & & a_{nn}^{(n)} \end{pmatrix}$$

cost : $O(n^3)$

$\det(A)$ by def.

$$= \sum_{\sigma : \text{permutations of } \{1,\cdots,n\}} (-1)^{\sigma} \prod a_{i,\sigma(i)}$$

cost : $O(n \cdot n!)$