Recall Gaussian elimination

$$
\begin{array}{c}
E_1 \\ \vdots \\ E_i \\ E_{i+1} \\ \vdots \\ E_n
\end{array}
\left(
\begin{array}{cccc|c}
a_{11} & \cdots & \cdots & a_{1n} & b_1 \\
 & \ddots & & \vdots & \vdots \\
 & & a_{ii} & \cdots\ a_{in} & b_i \\
 & 0 & a_{i+1,i} & \cdots\ a_{i+1,n} & b_{i+1} \\
 & & \vdots & \vdots & \vdots \\
 & & a_{ni} & \cdots\ a_{nn} & b_n
\end{array}
\right)
$$

Step i : $E_j \rightsquigarrow E_j - \dfrac{a_{ji}}{a_{ii}} E_i$

$j = i+1, \cdots, n$

Computational cost of step i : $O((n-i)^2)$

Total computational cost of G. e.:

$$O\left((n-1)^2+(n-2)^2+\dots+1^2\right) = O(n^3)$$

Backward substitution:

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij}\, x_j\right)$$

$i = n, n-1, \dots, 1$

comp. cost: $O(n^2)$

6.2 Pivoting strategy

Issue: what if $a_{ii} = 0$?

- Find $k \in \{i+1, \dots, n\}$ s.t. $a_{ki} \neq 0$

  Exchange E_i and E_k

  (there is always some $a_{ki} \neq 0$, otherwise A is not invertible)

- Due to round-off error, it's better to do this even when $a_{ii} \neq 0$ but small.

Example : 3 digit decimal machine number, rounding

$$\begin{array}{c} E_1 \\ E_2 \end{array} \left(\begin{array}{cc|c} 0.001 & 2 & 2 \\ -1 & 3 & 0 \end{array} \right)$$

$$\Big\downarrow \quad E_2 \rightsquigarrow E_2 + 1000 E_1$$

$$\left(\begin{array}{cc|c} 0.001 & 2 & 2 \\ 0 & \underbrace{2003}_{fl \,\to\, 2000} & 2000 \end{array} \right)$$

$$x_2 = \frac{2000}{2000} = 1$$

$$x_1 = \frac{1}{0.001}(2 - 2x_2) = 0$$

exact sol'n $x_2 \approx 1$, $x_1 \approx 3$

Gaussian elimination w/ partial pivoting

Step i : Choose k s.t. $|a_{ki}| = \max_{i \le j \le n} |a_{ji}|$

Then exchange E_i and E_k

Then as before:

$$E_j \rightsquigarrow E_j - \frac{a_{ji}}{a_{ii}} E_i \qquad j = i+1, \dots, n$$

Extra comp. cost (total): $O(n^2)$

Diagonally dominant matrices (part of sec. 6.6)

Def $n \times n$ matrix $A = (a_{ij})$ is

diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i = 1, \dots, n$$

strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i = 1, \dots, n$$

$$\begin{pmatrix} \Box & \scriptstyle\square & \scriptstyle\square & \scriptstyle\square \\ \scriptstyle\square & \Box & \scriptstyle\square & \scriptstyle\square \\ \scriptstyle\square & \scriptstyle\square & \Box & \scriptstyle\square \\ \scriptstyle\square & \scriptstyle\square & \scriptstyle\square & \Box \end{pmatrix}$$

<u>Ex</u>

$\begin{pmatrix} 3 & -1 \\ 0.1 & 3 \end{pmatrix}$ is s. d. d.

$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ is d. d. but not s. d.d.

$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 1 \\ -2 & -3 & 4 \end{pmatrix}$ is not d. d.

Thm A s.d.d. matrix is invertible.

When doing G.e., every intermediate stage is also s.d.d.

(in particular, no pivoting is needed).

Pf of "invertible"

Pf by contradiction: Suppose $A = (a_{ij})$ is s.d.d. and non-invertible.

Then $\exists \vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \vec{0}$

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad i = 1, \cdots, n$$

s.d.d. $\quad |a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i=1,\ldots,n$

Let k be the index s.t. $|x_k| = \max_i |x_i| > 0$

Then $\displaystyle\sum_{j=1}^{n} a_{kj} x_j = 0$

$$|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$$

$$a_{kk} x_k = - \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j$$

However,

$$\left| - \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{kj} x_j \right| \leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| \cdot |x_j|$$

$$\leq \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| \cdot |x_k|$$

$$< |a_{kk}| \cdot |x_k| \qquad \text{contradiction!}$$

Invertibility may fail if not "strict":

example: $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$

related concept: irreducible d.d.