AMSC/CMSC 460 — Midterm Exam 2, version A, solution
7 Nov 2019, Instructor: Ruiwen Shu

Total time: 75 minutes.

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators in this exam.

Problem 1. (20 points) Solve the following linear system of equations using Gaussian elimination **with partial pivoting strategy**.

$$\begin{cases} x_1 - x_2 + 2x_3 = 3 \\ 2x_1 + 6x_2 - x_3 = -1 \\ -x_1 - 5x_2 + 3x_3 = 2 \end{cases}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 6 & -1 & -1 \\ -1 & -5 & 3 & 2 \end{array}\right)$$

Exchange row 1 and row 2 (by pivoting):

$$\left(\begin{array}{ccc|c} 2 & 6 & -1 & -1 \\ 1 & -1 & 2 & 3 \\ -1 & -5 & 3 & 2 \end{array}\right)$$

Use row 1 to do Gaussian elimination

$$\left(\begin{array}{ccc|c} 2 & 6 & -1 & -1 \\ 0 & -4 & \frac{5}{2} & \frac{7}{2} \\ 0 & -2 & \frac{5}{2} & \frac{3}{2} \end{array}\right)$$

No need to exchange rows. Use row 2 to do Gaussian elimination

$$\left(\begin{array}{ccc|c} 2 & 6 & -1 & -1 \\ 0 & -4 & \frac{5}{2} & \frac{7}{2} \\ 0 & 0 & \frac{5}{4} & -\frac{1}{4} \end{array}\right)$$

Then backward substitution:

$$x_3 = -\frac{1}{4}/\frac{5}{4} = -\frac{1}{5}$$

$$x_2 = (\frac{7}{2} - \frac{5}{2}x_3)/(-4) = -1$$

$$x_1 = (-1 - 6x_2 + x_3)/2 = \frac{12}{5}$$

Problem 2. (15 points) Determine whether the following matrix is diagonally dominant, and whether it is positive definite. Justify your answer.

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & -4 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

It is diagonally dominant: on row 1:

$$|4| > |-1| + |-1|$$

on row 2:

$$|-4| > |-1| + |2|$$

on row 3:

$$|4| > |-1| + |2|$$

It is not positive definite because the second leading principle submatrix

$$det \begin{pmatrix} 4 & -1 \\ -1 & -4 \end{pmatrix} = -17 < 0$$

Problem 3. (15 points) Let $f(x) = \frac{1}{x+1}$. Find the polynomial $P(x)$ of degree no more than 1 such that $\sum_{k=1}^{3}(f(k) - P(k))^2$ is minimize. **You could write down the equations for the coefficients of $P(x)$, without solving them.**
There are three base points $x_1 = 1, x_2 = 2, x_3 = 3$ for this discrete least square problem. Write $P(x) = a_0 + a_1 x$. Then a_0, a_1 satisfies the linear system

$$\begin{pmatrix} \sum x_i^0 & \sum x_i^1 \\ \sum x_i^1 & \sum x_i^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \sum f(x_i) x_i^0 \\ \sum f(x_i) x_i^1 \end{pmatrix}$$

$$\sum x_i^0 = 3, \quad \sum x_i^1 = 6, \quad \sum x_i^2 = 14$$

$$\sum f(x_i) x_i^0 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}, \quad \sum f(x_i) x_i^1 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$$

Therefore

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \frac{13}{12} \\ \frac{23}{12} \end{pmatrix}$$

Problem 4. (20 points) Suppose you are given the following point values of f:

x	0	0.2	0.4	0.6
$f(x)$	10.6	10.3	9.5	8.2

Find the approximations of $f'(0.2)$ by using forward difference formula and three-point mid-point formula. Which result is more likely to have smaller error? Explain your answer.
Forward difference formula:

$$f'(0.2) \approx \frac{f(0.4) - f(0.2)}{0.2} = \frac{9.5 - 10.3}{0.2} = -4$$

Three-point midpoint formula:

$$f'(0.2) \approx \frac{f(0.4) - f(0)}{0.4} = \frac{9.5 - 10.6}{0.4} = -2.75$$

The result from midpoint formula is more likely to have smaller error because its order of accuracy (which is two) is higher than that of the forward difference formula (which is one).

Problem 5. (30=15+15 points)
(1) Approximate $\int_1^{1.5} \ln x \, \mathrm{d}x$ by Simpson rule, and estimate its error (using the error bound).
$h = \frac{1.5-1}{2} = 0.25$. $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$. $f(x) = \ln x$.

$$\int_1^{1.5} \ln x \, \mathrm{d}x = \frac{0.25}{3}(\ln 1 + 4 \cdot \ln 1.25 + \ln 1.5)$$

The error bound is

$$\frac{1}{90} h^5 \sup_{x \in [1,1.5]} |f^{(4)}(x)|$$

$f^{(4)}(x) = -\frac{6}{x^4}$, and the supremum of $|f^{(4)}(x)|$ is achieved at $x = 1$. Therefore the error bound is

$$\frac{1}{90} 0.25^5 \cdot 6$$

(2) Consider a quadrature

$$\int_0^\infty f(x) e^{-2x} \, \mathrm{d}x \approx c_1 f(x_1)$$

Find c_1 and x_1 such that this quadrature has degree of accuracy at least 1.
By definition of degree of accuracy, the quadrature needs to be exact for $f(x) = 1, x$. This gives

$$c_1 = \int_0^\infty 1 \cdot e^{-2x} = \frac{1}{2}$$

$$c_1 x_1 = \int_0^\infty x \cdot e^{-2x} = \frac{1}{4}$$

(by integration by parts) which gives $x_1 = \frac{1}{2}$.