AMSC/CMSC 460 Midterm Exam 1, version A, solution
26 Sept 2019, Instructor: Ruiwen Shu

Total time: 75 minutes.

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators in this exam.
(4) For Problem 2, keep at least 5 effective digits in all your intermediate results.
(5) For Problem 3, you do not need to simplify your result.

Problem 1. (10=5+5 points) Using decimal machine numbers with 3 effective digits, compute $\frac{10}{3}\times 6$ by using
(1) chopping.
$\frac{10}{3}$ becomes 0.333×10^1. Then $(0.333\times 10^1)\times 6 = 0.1998\times 10^2$ which truncates to 0.199×10^2 by chopping.
(2) rounding.
$\frac{10}{3}$ becomes 0.333×10^1. Then $(0.333\times 10^1)\times 6 = 0.1998\times 10^2$ which truncates to 0.200×10^2 by rounding.

Problem 2. (30=15+15 points) Let $f(x)=\sin x+0.1$, and we want to find the root of f near $x=3$.
(1) Using Newton's method with $p_0=3$, compute the approximations p_1 and p_2.
Newton's method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})},\quad f(x)=\sin x+0.1,\quad f'(x)=\cos x$$

$$p_1 = 3 - \frac{\sin(3)+0.1}{\cos(3)} = 3.2436$$

$$p_2 = 3.2436 - \frac{\sin(3.2436)+0.1}{\cos(3.2436)} = 3.2418$$

(2) Using secant method with $p_0=3$ and $p_1=3.5$, compute the approximation p_2.
Secant method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1}-p_{n-2})}{f(p_{n-1})-f(p_{n-2})}$$

$$p_2 = 3.5 - \frac{(\sin(3.5)+0.1)(3.5-3)}{(\sin(3.5)+0.1)-(\sin(3)+0.1)} = 3.2451$$

Problem 3. (20=10+10 points) Let $f(x)=e^{-x}$.
(1) Compute its Lagrange interpolation $P(x)$ at base points $0,1,2$.

$$x_0=0,\ x_1=1,\ x_2=2$$

$$P(x) = L_0(x)f(x_0)+L_1(x)f(x_1)+L_2(x)f(x_2)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)}$$

$$L_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)}$$

$$L_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)}$$

$$P(x) = \frac{(x-1)(x-2)}{2} \cdot 1 + \frac{x(x-2)}{-1} \cdot e^{-1} + \frac{x(x-1)}{2} \cdot e^{-2}$$

(2) Estimate the error $|f(0.5) - P(0.5)|$. (Use the error bound. **Do not** compute $P(0.5)$ explicitly!)
The theorem for error estimate, applied to the interval $(0, 2)$, says

$$f(0.5) - P(0.5) = \frac{f^{(3)}(\xi)}{3!}(0.5-0)(0.5-1)(0.5-2)$$

for some $\xi \in (0, 2)$. To estimate $f^{(3)}(\xi)$, notice that $f^{(3)}(x) = -e^{-x}$. Then $|f^{(3)}(x)| = e^{-x}$ which achieves its maximum on $[0, 2]$ at $x = 0$, since $x \mapsto e^{-x}$ is a decreasing function. Therefore

$$|f^{(3)}(\xi)| \le e^{-0} = 1, \quad \xi \in (0, 2)$$

and we conclude

$$|f(0.5) - P(0.5)| \le \frac{1}{6} 0.5 \cdot 0.5 \cdot 1.5$$

Problem 4. (20 points) Let $g(x)$ be a C^2 function defined on $\mathbb{R}$, and p be a fixed point of g. Assume $g'(p) = \frac{1}{2}$ and $|g''(x)| \le 100$, $\forall x \in \mathbb{R}$. Give an explicit value of $\delta > 0$ such that the fixed point iteration converges to p for any initial value $p_0 \in [p-\delta, p+\delta]$. Justify your conclusion.
Take $\delta = \frac{1}{400}$. Then by mean value theorem applied to g',

$$g'(x) - g'(p) = g''(\xi)(x-p)$$

for some ξ between x and p. Therefore, if $x \in [p-\delta, p+\delta]$, then

$$|g'(x) - g'(p)| = |g''(\xi)| \cdot |x-p| \le 100\delta = \frac{1}{4}$$

Since $g'(p) = \frac{1}{2}$, this means $\frac{1}{4} \le g'(x) \le \frac{3}{4}$, or in particular,

$$|g'(x)| \le \frac{3}{4}, \quad \forall x \in [p-\delta, p+\delta]$$

Then we claim that g maps $[p-\delta, p+\delta]$ into itself. In fact, if $x \in [p-\delta, p+\delta]$, then by mean value theorem applied to g and using the fact that p is a fixed point of g,

$$g(x) - p = g(x) - g(p) = g'(\xi_1)(x-p)$$

for some ξ_1 between x and p, and in particular, in the interval $[p-\delta, p+\delta]$. Therefore

$$|g(x) - p| = |g'(\xi_1)| \cdot |x - p| \leq \frac{3}{4}\delta$$

which shows $g(x) \in [p-\delta, p+\delta]$.
Now we can apply the fixed point theorem for g on $x \in [p-\delta, p+\delta]$ to conclude that the fixed point iteration with initial value in this interval converges to p.
Remark: Any $0 < \delta < \frac{1}{200}$ will make the same proof work.

Problem 5. (20 points) Let $f(x)$ be a polynomial of degree 20, and $x_0, x_1, \ldots, x_{21} \in \mathbb{R}$ be distinct points. Show that the divided difference

$$f[x_0, \ldots, x_{21}] = 0$$

Since f itself is a polynomial of degree no more than 21 and agrees with f at $x_0, \ldots, x_{21}$, it is the Lagrange interpolation of itself at $x_0, \ldots, x_{21}$. Using the Newton's form

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \ldots, x_{21}](x - x_0)\cdots(x - x_{20})$$

Compare the degree-21 coefficients on both sides: LHS is 0 since f has degree 20; RHS is $f[x_0, \ldots, x_{21}]$ since the last term is the only one with a degree-21 term.
Remark: There might be other methods of solving this problem, but the above method seems to be the fastest, as far as I know.