AMSC/CMSC 460 — MIDTERM EXAM 1 SOLUTION

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Total time: 75 minutes.

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators in this exam.
(4) For Problem 2, keep at least 3 effective digits in all your intermediate results.
(5) For Problem 4, you do not need to simplify your result.

Problem 1. (10=5+5 points) Using decimal machine numbers with 3 effective digits, compute $\frac{2}{3} \times 3$ by using
(1) chopping.
(2) rounding.

(1) chopping: $fl(\frac{2}{3}) = 0.666 \times 10^0$, $fl(3) = 0.3 \times 10^1$, $fl(fl(\frac{2}{3}) \times fl(3)) = fl(0.1998 \times 10^1) = 0.199 \times 10^1$.
(2) rounding: $fl(\frac{2}{3}) = 0.667 \times 10^0$, $fl(3) = 0.3 \times 10^1$, $fl(fl(\frac{2}{3}) \times fl(3)) = fl(0.2001 \times 10^1) = 0.200 \times 10^1$.

Problem 2. (35=10+10+10+5 points) Let $f(x) = x^3 - 2$, and we try to approximate its root.

(1) Justify that one could apply the bisection method to $f(x)$ on $[1, 1.4]$. Apply it twice to find p_3 as an approximation of a root of f.

(2) Using the bisection method as in (1), to guarantee that the error $|p_n - p|$ is less than 10^{-2}, what is the smallest n needed?

(3) Using Newton's method with $p_0 = 1$, compute the approximation p_2.

(4) Comparing your approximations from (1) and (3), which is likely to be more accurate? Explain your answer.

(1) $f(1) = -1 < 0$, $f(1.4) = 1.4^3 - 2 = 0.744 > 0$, $f(1)f(1.4) < 0$. Therefore one can apply bisection.
$p_1 = 1.2$, $f(p_1) = -0.272 < 0$.
$[a_2, b_2] = [1.2, 1.4]$, $p_2 = 1.3$, $f(p_2) = 0.197 > 0$.
$[a_3, b_3] = [1.2, 1.3]$, $p_3 = 1.25$.
(2) Error bound $|p_n - p| \le \frac{b-a}{2^n} = \frac{0.4}{2^n}$. To make $\frac{0.4}{2^n} \le 10^{-2}$, $2^n \ge 40$, the smallest possible n is 6.
(3) $f'(x) = 3x^2$.
$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{1^3-2}{3\times 1^2} = \frac{4}{3} \approx 1.33$.
$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 1.33 - \frac{1.33^3-2}{3\times 1.33^2} \approx 1.26$.

(4) The approximation in (3) is likely to be more accurate, because Newton's method has quadratic convergence, which is faster than linear convergence for bisection method. (I also give full credit to those who checked that the true error of (3) is smaller than that of (1), or checked the same thing with $|p_n^3 - 2|$.)

Problem 3. (15 points) Let $f(x) = e^x - 2$. f has a unique root p in $[0, 1]$. Find a function $g(x)$, so that the fixed point iteration applied to g with any initial point $p_0 \in [0, 1]$ converges to p. Justify your answer (you are **NOT** required to justify the condition $g(x) \in [0, 1]$, $\forall x \in [0, 1]$).

(The answer to this problem is not unique.) Take $g(x) = x - \frac{f(x)}{10}$. Then clearly $f(x) = 0$ is equivalent to $g(x) = x$. To check the condition $|g'(x)| \leq k < 1$, $\forall x \in [0, 1]$ in Fixed Point Theorem, we first notice that $f'(x) = e^x$ is an increasing function, and thus

$$1 = f'(0) \leq f'(x) \leq f'(1) = e, \quad \forall x \in [0, 1]$$

Since $g'(x) = 1 - \frac{f'(x)}{10}$, we obtain

$$1 - \frac{e}{10} \leq g'(x) \leq 1 - \frac{1}{10}, \quad \forall x \in [0, 1]$$

Notice that $1 - \frac{e}{10} > 0$. Therefore we obtain the condition $|g'(x)| \leq k < 1$, $\forall x \in [0, 1]$ with $k = \frac{9}{10} < 1$.

Problem 4. (25=10+15 points) Let $f(x) = \ln(1 + x)$.

(1) Compute its Lagrange interpolation $P(x)$ at base points $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, using Lagrange interpolating polynomials.

(2) Use the error bound to estimate the error $|f(1) - P(1)|$.

(1)

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= \ln 1 \cdot \frac{(x-2)(x-3)}{(0-2)(0-3)} + \ln 3 \cdot \frac{(x-0)(x-3)}{(2-0)(2-3)} + \ln 4 \cdot \frac{(x-0)(x-2)}{(3-0)(3-2)}$$

(2) The error estimate formula gives

$$f(1) - P(1) = \frac{f'''(\xi)}{3!}(1-0)(1-2)(1-3)$$

for some $\xi \in (0, 3)$. $f'''(x) = \frac{2}{(1+x)^3}$. Therefore

$$|f'''(\xi)| \leq |f'''(0)| = 2$$

Therefore

$$|f(1) - P(1)| \leq \frac{2}{6} \cdot 1 \cdot 1 \cdot 2 = \frac{2}{3}$$

Problem 5. (15 points) Let $f(x) = x^3 - 5x^2 + 12x - 2$, and $x_k = k$, $k = 0, 1, 2, 3$. Compute the divided difference $f[x_0, x_1, x_2, x_3]$.

Method 1: use a theorem in class:

$$f[x_0, x_1, x_2, x_3] = \frac{f'''(\xi)}{3!}$$

for some $\xi \in (0, 3)$. Since $f'''(x) = 6$ is a constant function, we obtain $f[x_0, x_1, x_2, x_3] = 1$.
Method 2: direct calculation:

$$f[x_0] = -2$$

$$f[x_1] = 6, \quad f[x_0, x_1] = \frac{6-(-2)}{1-0} = 8$$

$$f[x_2] = 10, \quad f[x_1, x_2] = \frac{10-6}{2-1} = 4, \quad f[x_0, x_1, x_2] = \frac{4-8}{2-0} = -2$$

$$f[x_3] = 16, \quad f[x_1, x_2] = \frac{16-10}{3-2} = 6, \quad f[x_1, x_2, x_3] = \frac{6-4}{3-1} = 1, \quad f[x_0, x_1, x_2, x_3] = \frac{1-(-2)}{3-0} = 1$$