AMSC/CMSC 460 — FINAL EXAM, VERSION A

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Total time: 120 minutes.

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name on every answer sheet.
(3) You are allowed to use calculators in this exam.

Problem 1. (15=12+3 points)
(1) Compute the LU decomposition of the following matrix (**without** row exchanges):

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 6 \\ 3 & -4 & 4 \end{pmatrix}$$

(2) Use the LU decomposition in (1) to compute the determinant of A.

Problem 2. (10=3+7 points)
(1) Justify that the bisection method can be applied to find a root of $f(x) = x^3 - x + 1$ on $[-2, 0]$.

(2) Conduct this bisection method and give an interval of length 0.5 containing a root of f.

Problem 3. (15 points) Find the orthonormal polynomials $\phi_0(x)$ and $\phi_1(x)$ on the interval $[1, 3]$ with weight function $w(x) = x$.

Problem 4. (10 points) Apply the Richardson extrapolation once, to improve the order of accuracy of the numerical differentiation formula

$$f''(x) \approx \frac{1}{h^2}(f(x+h) - 2f(x) + f(x-h))$$

Problem 5. (20=10+10 points)
(1) Apply the (forward) Euler's method with $h = 0.5$ to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y + e^t, \quad y(0) = 1$$

to approximate $y(1)$.

(2) Using the error bound, estimate the error of the result in (1), given the exact solution

$$y(t) = 2e^{2t} - e^t$$

More problems on the back!

Problem 6. (10 points) Apply the midpoint method with $h = 0.1$ to the ODE system

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = y_2 \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = -y_1^3 \end{cases} \qquad y_1(0) = 1,\ y_2(0) = -0.2$$

to approximate $y_1(0.1)$ and $y_2(0.1)$.

Problem 7. (10 points) For the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$, consider a multistep method

$$w_{i+1} = c_1 w_i + c_2 w_{i-2} + c_3 f(t_{i+1}, w_{i+1})$$

Compute c_1, c_2, c_3 so that this method has at least second order accuracy (in the sense of local truncation error).

Problem 8. (10=5+5 points)
(1) For the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$, consider the following Runge-Kutta method:

$$\begin{cases} w_i^{(1)} = w_i + 0.7hf(t_i + 0.7h, w_i^{(1)}) \\ w_i^{(2)} = w_i + 0.3hf(t_i + 0.7h, w_i^{(1)}) + 0.7hf(t_i + h, w_i^{(2)}) \\ w_{i+1} = w_i^{(2)} \end{cases}$$

Compute its growth factor $Q(h\lambda)$.

(2) When $h\lambda$ is a negative real number with $|h\lambda|$ being very large, is it inside the stability region of the R-K method in (1)? Either explain your answer or give rigorous justification.