# 6.1 Linear systems of equations

## Gaussian elimination

**Question**

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Know $\{a_{ij}\}$, $\{b_i\}$, want $\{x_i\}$.

In matrix form: $Ax = b$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

**Ex** Solve
$$\begin{cases} x_1 - 2x_2 + x_3 = -1 & \text{---- } E_1 \\ 2x_1 - x_2 - x_3 = 2 & \text{---- } E_2 \\ -x_1 - 4x_2 + x_3 = 3 & \text{---- } E_3 \end{cases}$$

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$E_1$ $\quad x_1 - 2x_2 + x_3 = -1$ ---- $E_1$

$E_2 + (-2)E_1$ $\quad 3x_2 - 3x_3 = 4$ ---- $E_2$

$E_3 + 1\cdot E_1$ $\quad -6x_2 + 2x_3 = 2$ ---- $E_3$

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$E_1$ $\quad x_1 - 2x_2 + x_3 = -1$

$E_2$ $\quad 3x_2 - 3x_3 = 4$

$E_3 + 2\cdot E_2$ $\quad -4x_3 = 10$

$$x_3 = \frac{10}{-4} = -\frac{5}{2}$$

$$x_2 = \frac{1}{3}(4 + 3x_3) = \frac{1}{3}\left(4 + 3\cdot\left(-\frac{5}{2}\right)\right) = -\frac{7}{6}$$

$$x_1 = -1 + 2x_2 - x_3 = -1 + 2\cdot\left(-\frac{7}{6}\right) - \left(-\frac{5}{2}\right) = -\frac{5}{6}$$

In matrix form

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 2 & -1 & -1 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right)$$

"augmented matrix"

$E_2 \rightsquigarrow E_2 + (-2) E_1$

$E_3 \rightsquigarrow E_3 + 1 \cdot E_1$

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & -6 & 2 & 2 \end{array}\right)$$

$E_3 \rightsquigarrow E_3 + 2 E_2$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & -4 & 10 \end{array}\right)$$

"upper triangular form"

# Gaussian elimination

Given $Ax = b$, write augmented matrix

$$\begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_n \end{array} \left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array} \right)$$

issue: $a_{11} = 0$?

Step 1:

$$E_j \rightsquigarrow E_j - \frac{a_{j1}}{a_{11}} E_1$$

$j = 2, \ldots, n$

$$\begin{array}{c} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_n \end{array} \left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ 0 & a_{32} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} & b_n \end{array} \right)$$

Here row 2 and below have been changed

Step 2:

$$E_j \rightsquigarrow E_j - \frac{a_{j2}}{a_{22}} E_2$$

$$j = 3, \ldots, n$$

$$\left( \begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ 0 & 0 & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} & b_n \end{array} \right)$$

Step $i$:

$$E_j \rightsquigarrow E_j - \frac{a_{ji}}{a_{ii}} E_i$$

$$j = i+1, \dots, n$$

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & & a_{1n} & b_1 \\ & a_{22} & \cdots & & a_{2n} & b_2 \\ & & \ddots & & \vdots & \vdots \\ & 0 & & a_{n-1,n-1} & a_{n-1,n} & b_{n-1} \\ & & & & a_{nn} & b_n \end{array}\right)$$

Backward substitution: get $x_n, x_{n-1}, \dots, x_1$ from upper ~~triangular~~ triangular form.

$$x_n = \frac{1}{a_{nn}} b_n$$

$$x_{n-1} = \frac{1}{a_{n-1,n-1}}\left(b_{n-1} - a_{n-1,n} x_n\right)$$

$$\vdots$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j\right)$$