3.3 Newton's divided difference

Let $P(x)$ be the Lagrange interp. of $f(x)$ at $x_0, \cdots, x_n$. Express $P(x)$ as

$$P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+ \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Know: $P(x_j) = f(x_j)$, $j = 0, 1, \cdots, n$

$j=0$: $a_0 = f(x_0)$

$j=1: \quad a_0 + a_1(x_1 - x_0) = f(x_1)$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$j=2: \quad a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

$= f(x_2)$

$$a_2 = \frac{f(x_2) - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot (x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + f(x_1) - f(x_0) - \frac{x_2 - x_0}{x_1 - x_0}(f(x_1) - f(x_0))}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right]$$

Observation: a_k only depends on

f and $x_0, \cdots, x_k$

Denote $a_k = f[x_0, \cdots, x_k]$

(called <u>divided difference</u>)

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, \cdots, x_n](x - x_0) \cdots (x - x_{n-1})$$

"Newton's form" of Lagrange interp.

or Newton interp.

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

<u>Thm</u> The divided difference $f[x_0, \cdots, x_k]$ are given iteratively by

$$f[x_0, \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

$$f[x_j] = f(x_j)$$

pf)

$$P_{0\cdots k}(x) = \frac{(x-x_0)\,P_{1\cdots k}(x) - (x-x_k)\,P_{0\cdots(k-1)}(x)}{x_k - x_0}$$

coeff. of highest degree $(\deg = k)$

$$f[x_0, \cdots, x_k] = \frac{1}{x_k - x_0}\Big[f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}] \Big]$$

Ex Compute Newton interpolation for

$f(x) = \frac{1}{x}$ at $x_0 = 3$, $x_1 = 5$, $x_2 = 6$

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$f[x_0] = \frac{1}{3}$

$f[x_1] = \frac{1}{5}$

$f[x_2] = \frac{1}{6}$

$$f[x_0, x_1] = \frac{\frac{1}{5} - \frac{1}{3}}{5 - 3} = -\frac{1}{15}$$

$$f[x_1, x_2] = \frac{\frac{1}{6} - \frac{1}{5}}{6 - 5} = -\frac{1}{30}$$

$$f[x_0, x_1, x_2] = \frac{-\frac{1}{30} - \left(-\frac{1}{15}\right)}{6-3} = \frac{1}{90}$$

$$P(x) = \frac{1}{3} + \left(-\frac{1}{15}\right)(x-3) + \frac{1}{90}(x-3)(x-5)$$

Advantage of Lagrange form:

- Formula is explicit

- Symmetric with respect to $\{x_k\}$

Advantage of Newton form

- Easy to add points
  (similar to Neville's method)

Thm Let $f \in C^n[a,b]$, $x_0, \dots, x_n \in [a,b]$ distinct. Then

$$f[x_0, \dots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some $\xi \in (a,b)$

- can be used to approximate $f^{(n)}(x)$ when $[a,b]$ is short (numerical differentiation).

- $f[x_0, \cdots, x_n] = 0$ if f is a poly. of deg $\leq n-1$

- An explicit formula for div. diff.

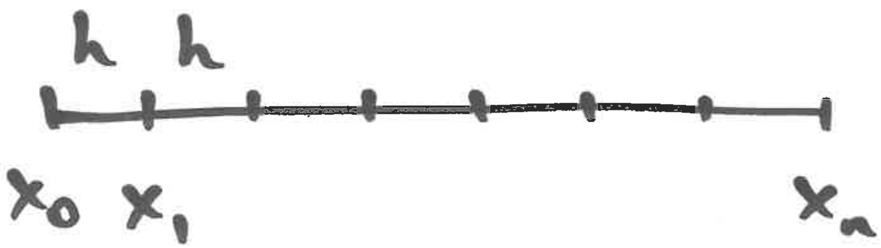
$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \cdots, x_n](x - x_0) \cdots (x - x_{n-1})$$

$$= \sum_{k=0}^{n} L_k(x) f(x_k), \qquad L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}$$

coeff. of highest degree (deg = n)

$$f[x_0, \cdots, x_n] = \sum_{k=0}^{n} f(x_k) \cdot \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{1}{x_k - x_j}$$

- When $x_0, \cdots, x_n$ have equal spacing



$x_k = x_0 + kh$, $k = 0, \cdots, n$

for some $h > 0$.

$$\prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{1}{x_k - x_j} = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{1}{(x_0 + kh) - (x_0 + jh)}$$

$$= \frac{1}{h^n} \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{1}{k-j}$$

$$= \frac{1}{h^n} \prod_{j=0}^{k-1} \frac{1}{k-j} \cdot \prod_{j=k+1}^{n} \frac{1}{k-j}$$

$$= \frac{1}{h^n} \cdot \frac{1}{k!} \cdot \frac{1}{(n-k)!} (-1)^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{1}{h^n \cdot n!} \binom{n}{k} \cdot (-1)^{n-k}$$

$$f[x_0, \cdots, x_n] = \frac{1}{h^n \cdot n!} \sum_{k=0}^{n} f(x_k) \binom{n}{k} (-1)^{n-k}$$