Recall: Lagrange interp. of f

at $x_0, x_1, \cdots, x_n$ is

$$P(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$$

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}$$

Thm (error estimate of Lagrange interp.)

Let $x_0, x_1, \cdots, x_n \in [a,b]$ be distinct, $f \in C^{n+1}[a,b]$. Then, for any $x \in [a,b]$ $\exists\ \xi = \xi(x) \in (a,b)$ s.t.

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Is $|(x-x_0)(x-x_1)\cdots(x-x_n)|$ small ?

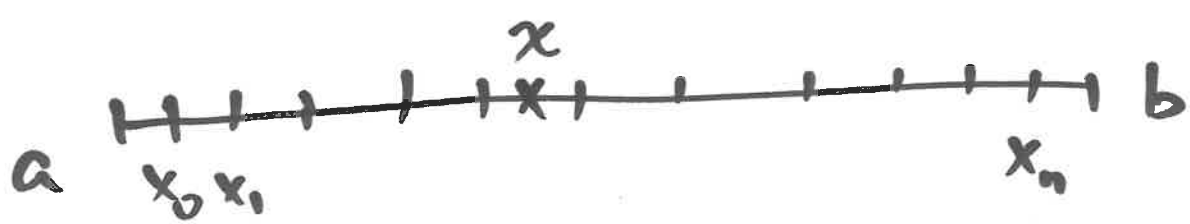
- If $[a,b]$ is very short, then

  $|x-x_j| \le b-a$ is small

  $\Rightarrow |\cdots\cdots|$ is very small

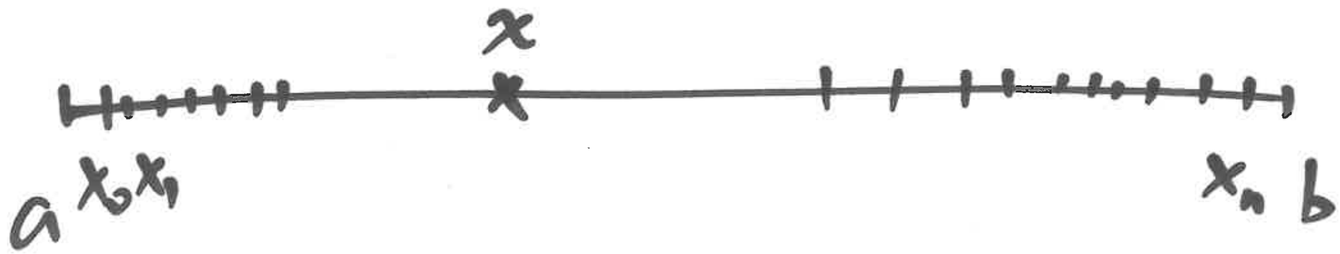
  (similar to Taylor expansions)

- If $[a,b]$ is not short, but $\{x_j\}$ are distributed almost uniformly



then there are always some small $|x-x_j|$

- If there are big gaps in $\{x_j\}$



then $| \cdots |$ can be large.

- Sometimes $|f^{(n+1)}(\xi)|$ can be large.

- Lagrange interp. doesn't necessarily give the poly. approx. stated in Weierstrass thm.

Ex Find Lagrange interp. of $f(x) = \frac{1}{x}$

at $x_0 = 3$, $x_1 = 5$, $x_2 = 6$

and estimate error at $x = 4$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-5)(x-6)}{(3-5)(3-6)}$$

$$L_1(x) = \frac{(x-3)(x-6)}{(5-3)(5-6)}$$

$$L_2(x) = \frac{(x-3)(x-5)}{(6-3)(6-5)}$$

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$= \frac{1}{3} \cdot \frac{(x-5)(x-6)}{(3-5)(3-6)} + \frac{1}{5} \cdot \frac{(x-3)(x-6)}{(5-3)(5-6)}$$

$$+ \frac{1}{6} \cdot \frac{(x-3)(x-5)}{(6-3)(6-5)}$$

$$= \frac{1}{90} x^2 - \frac{7}{45} x + \frac{7}{10}$$

error estimate:

$$f(4) - P(4) = \frac{f^{(3)}(\xi)}{3!} \cdot (4-3)(4-5)(4-6)$$

$\xi \in (3,6)$

$$f(x) = \frac{1}{x} \qquad f^{(3)}(x) = -\frac{6}{x^4}$$

For $\xi \in (3,6)$, $\left|f^{(3)}(\xi)\right| \le \frac{6}{3^4}$

$$\left|f(4) - P(4)\right| \le \frac{6}{3^4} \cdot \frac{1}{6} \cdot 1 \cdot 1 \cdot 2 = \frac{2}{81} \approx 0.0247$$

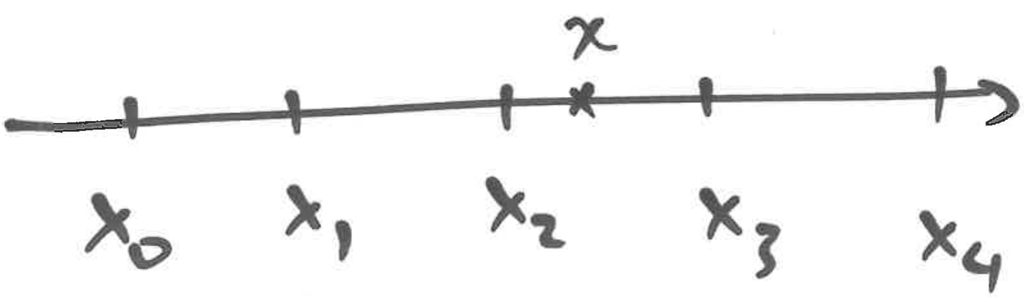
true error $\left|f(4) - P(4)\right| \approx 0.0056$

3.2 Neville's method

In reality, you often don't know how many points to use in Lagrange interp. to achieve certain accuracy.

$\rightsquigarrow$ Add pts one by one until seeing convergence.

- Computational cost of Lagrange interp. at $x_0, \dots, x_n$ at one value x is $O(n^2)$



Denote $P_{m_1 \dots m_k}$ as the Lagrange interp. at $x_{m_1}, \dots, x_{m_k}$

$$
\begin{tikzcd}[arrows=dash]
P_0 \arrow[dr, -to] & & & & \\
P_1 \arrow[r] \arrow[dr] & P_{01} \arrow[dr] & & & \\
\boxed{P_2} \arrow[r] \arrow[dr] & P_{12} \arrow[r] \arrow[dr] & P_{012} \arrow[dr] & & \\
\boxed{P_3} \arrow[r] \arrow[dr] & \boxed{P_{23}} \arrow[r] \arrow[dr] & \boxed{P_{123}} \arrow[r] \arrow[dr] & P_{0123} \arrow[dr] & \\
P_4 \arrow[r] & P_{34} \arrow[r] & \boxed{P_{234}} \arrow[r] & \boxed{P_{1234}} \arrow[r] & \boxed{P_{01234}}
\end{tikzcd}
$$

- Direct comp. cost of all "$\square$" is $O(n^3)$
- Neville's method: $O(n^2)$

Thm

$$P_{01\cdots k}(x) = \frac{(x-x_k)P_{01\cdots(k-1)}(x) - (x-x_0)P_{1\cdots k}(x)}{x_0 - x_k}$$

Pf) Call RHS $= \phi(x)$. We check defining properties of $P_{01\cdots k}(x)$

① $\deg \phi \le k$

In fact, $\deg P_{01\cdots(k-1)} \le k-1$

$\deg P_{1\cdots k} \le k-1$ ✓

② $\phi(x_j) = f(x_j)$, $j = 0, 1, \cdots, k$

- $$\phi(x_0) = \frac{\cancel{(x_0 - x_k)}\, P_{01\cdots(k-1)}(x_0)}{\cancel{x_0 - x_k}} = f(x_0)$$

- $\phi(x_k)$ similar

- $$\phi(x_j) = \frac{(x_j - x_k)\, P_{01\cdots(k-1)}(x_j) - (x_j - x_0)\, P_{1\cdots k}(x_j)}{x_0 - x_k}$$

$j = 1, \cdots, k-1$

$$= \frac{(x_j - x_k)\, f(x_j) - (x_j - x_0)\, f(x_j)}{x_0 - x_k}$$

$$= f(x_j)$$