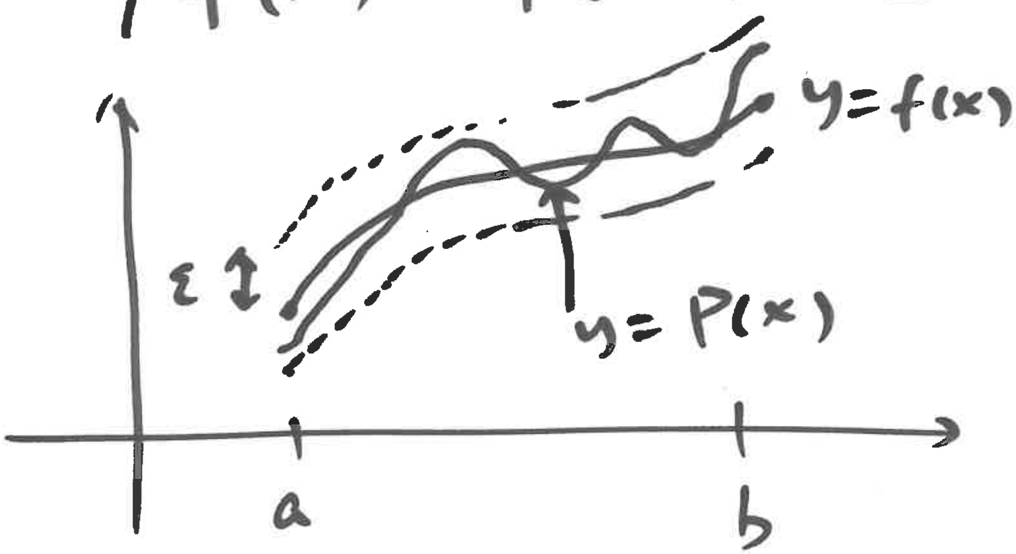
3.1 Lagrange interpolation

Weierstrass approximation theorem

Let $f \in C[a,b]$. For any $\varepsilon > 0$, $\exists$ polynomial $P(x)$ s.t.

$$|f(x) - P(x)| < \varepsilon \quad , \forall x \in [a,b]$$



- Polynomials are easier to handle than general functions (derivative, integral, ...)

<u>Question</u>: how to construct a poly. $P(x)$ to approximate $f(x)$?

- Taylor polynomial:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

good when $|x-x_0|$ is small, but can be bad when $|x-x_0|$ is large.

Ex $f(x) = \frac{1}{x}$ $\quad x_0 = 1$

$$f^{(k)}(x) = (-1)^k \cdot k! \cdot \frac{1}{x^{k+1}}$$

$$f^{(k)}(1) = (-1)^k \cdot k!$$

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k \cdot k!}{k!} \cdot (x-1)^k = \sum_{k=0}^{n} (-1)^k \cdot (x-1)^k$$

At $x=3$, $f(3) = \frac{1}{3}$ $\quad R = 1$

$$P_n(3) = \sum_{k=0}^{n} (-2)^k = \frac{1-(-2)^{n+1}}{1-(-2)}$$

the error $|f(3) - P_n(3)| \to \infty$ as $n \to \infty$

"radius of convergence" R of Taylor series: defined as when $|x - x_0| < R$, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \quad \text{converges.}$$

- Taylor poly. doesn't approximate $f(x)$ well for $|x - x_0| > R$.

- For many situations, only point values of f are available

(from measurements, numerical discretization, ...)

- If you are given, $f(x_0), f(x_1), \cdots, f(x_n)$, and want to approximate f by poly.

<u>Question</u> : find degree $\leq n$ poly. $P(x)$

s.t. $$P(x_k) = f(x_k), \quad k = 0, 1, \cdots, n$$

"Lagrange interpolation"

why $\deg \leq n$?

$$P(x) = \sum_{j=0}^{n} a_j x^j \quad , \ a_0, a_1, \cdots, a_n \text{ unknown.}$$

We have $$\sum_{j=0}^{n} a_j x_k^j = f(x_k) \ , \quad k = 0, 1, \cdots, n$$

a system of $n+1$ linear equations

in $n+1$ unknowns

$$\begin{pmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ & \cdots & & \\ x_n^0 & x_n^1 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$\hookrightarrow$ det $\neq 0$ if $x_0, x_1, \cdots, x_n$ are distinct

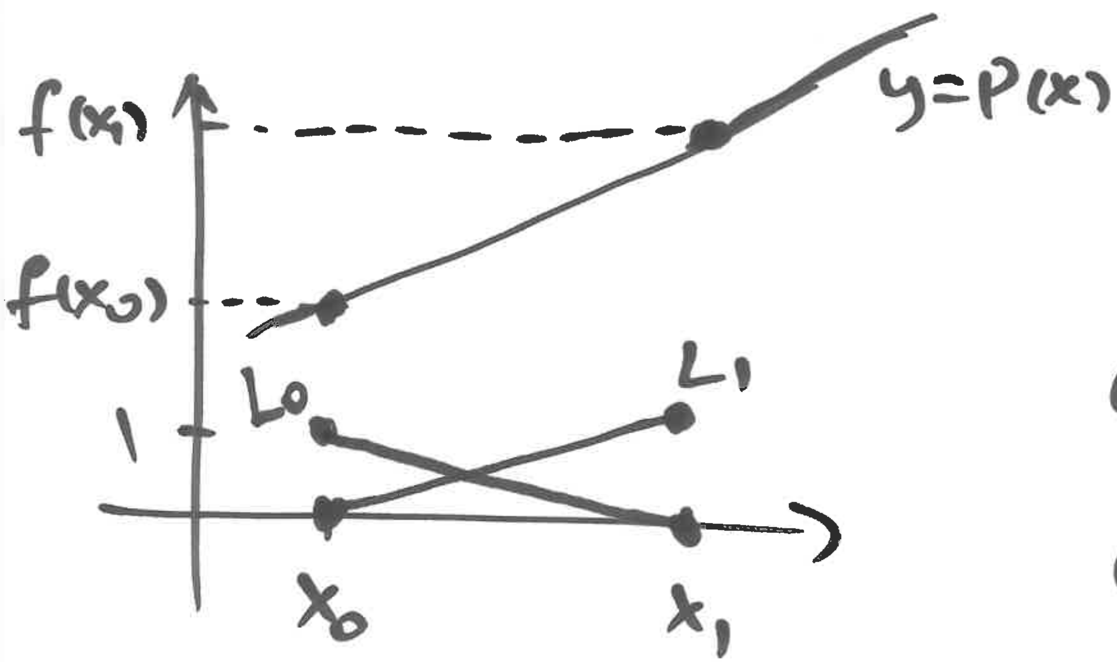
$\Rightarrow$ there exists a unique sol'n

"Vandermonde matrix"

Case $n=1$: given $f(x_0)$, $f(x_1)$

want $P(x) = a_0 + a_1 x$ s.t.

$$P(x_0) = f(x_0), \quad P(x_1) = f(x_1)$$



$$\begin{cases} a_0 + a_1 x_0 = f(x_0) \\ a_0 + a_1 x_1 = f(x_1) \end{cases}$$

$$a_1 (x_1 - x_0) = f(x_1) - f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$a_0 = f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0} x_0$$

$$P(x) = f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0} x_0 + \frac{f(x_1) - f(x_0)}{x_1 - x_0} x$$

$$= \left(1 + \frac{x_0}{x_1 - x_0} - \frac{x}{x_1 - x_0} \right) f(x_0)$$

$$+ \left(- \frac{x_0}{x_1 - x_0} + \frac{x}{x_1 - x_0} \right) f(x_1)$$

$$= \underbrace{\boxed{\frac{x - x_1}{x_0 - x_1}}}_{L_0(x)} f(x_0) + \underbrace{\boxed{\frac{x - x_0}{x_1 - x_0}}}_{L_1(x)} f(x_1)$$

$$L_0(x_0) = 1 \quad , \quad L_0(x_1) = 0$$

$$L_1(x_0) = 0 \quad , \quad L_1(x_1) = 1$$

General case: (base points $x_0, x_1, \cdots, x_n$)

define "<u>n-th Lagrange interpolating polynomial</u>"

$L_k(x)$ as the deg-n poly. s.t.

$$L_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$k = 0, 1, \cdots, n$

Then the Lagrange interpolation

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + \cdots + L_n(x) f(x_n)$$

$$= \sum_{j=0}^{n} L_j(x) f(x_j)$$

Check: $$P(x_k) = \sum_{j=0}^{n} \underbrace{L_j(x_k)}_{\begin{cases} =0 & \text{when } j \neq k \\ =1 & \text{when } j = k \end{cases}} f(x_j)$$

$$= 1 \cdot f(x_k) = f(x_k).$$

$$L_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}$$