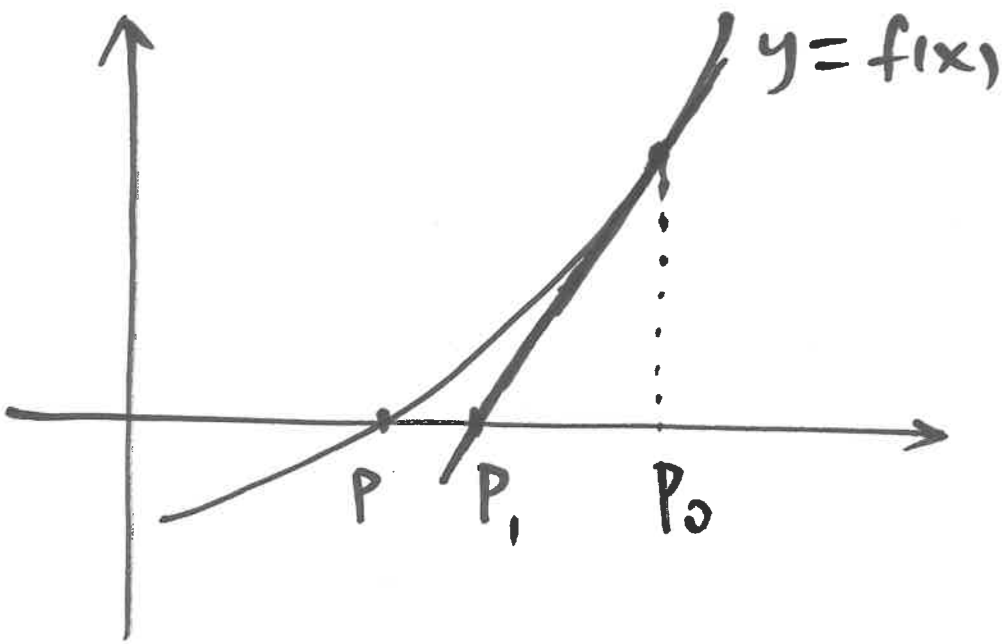
2.3, 2.4 Generalizations of Newton's method

Recall Newton's method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$



<u>Question</u>: how to avoid calculating f'?

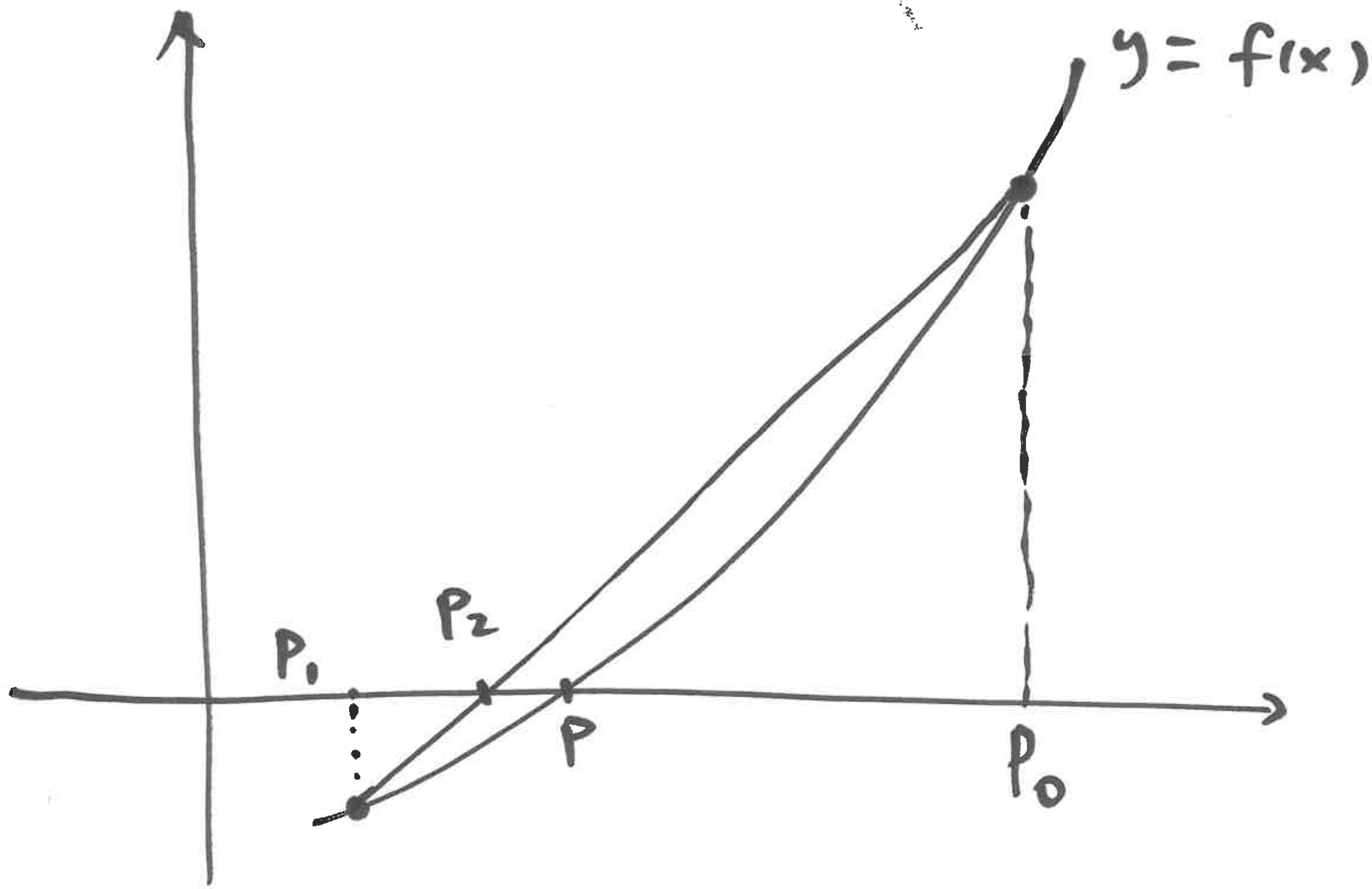
$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(\tilde{p})}{p_{n-1} - \tilde{p}}$$

for some $\tilde{p}$ near p_{n-1}.

- <u>The secant method</u>: take $\tilde{p} = p_{n-2}$

$$p_n = p_{n-1} - \frac{f(p_{n-1}) \cdot (p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \qquad n \geq 2$$

Start w/ p_0, p_1.



y = f(x)
p_2
p_1
p
p_0

- The convergence rate of secant method is faster than linear, slower than quadratic.

$$|p_{n+1} - p_n| \leq C |p_n - p_{n-1}|^{\alpha}$$

$$\alpha = \frac{\sqrt{5}+1}{2} \approx 1.6\cdots$$

Ex Use secant method (up to p_3) to approximate $\sqrt{2}$ by applying on $f(x) = x^2 - 2$ w/ $p_0 = 1$, $p_1 = 2$

$$p_2 = p_1 - \frac{f(p_1)\cdot(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{2\cdot(2-1)}{2-(-1)} = \frac{4}{3}$$

$$p_3 = p_2 - \frac{f(p_2)\cdot(p_2 - p_1)}{f(p_2) - f(p_1)} = \frac{4}{3} - \frac{\left(-\frac{2}{9}\right)\cdot\left(\frac{4}{3} - 2\right)}{\left(-\frac{2}{9}\right) - 2}$$

$$= \frac{7}{5}$$

Question: How to achieve quadratic Convergence for $f(x)$ w/ multiple roots?

Def A solution p of $f(x)=0$ is called a root (zero) of multiplicity m if $\exists\, q(x)$ s.t. $f(x) = (x-p)^m q(x)$ w/ $\lim_{x \to p} q(x) \neq 0$.

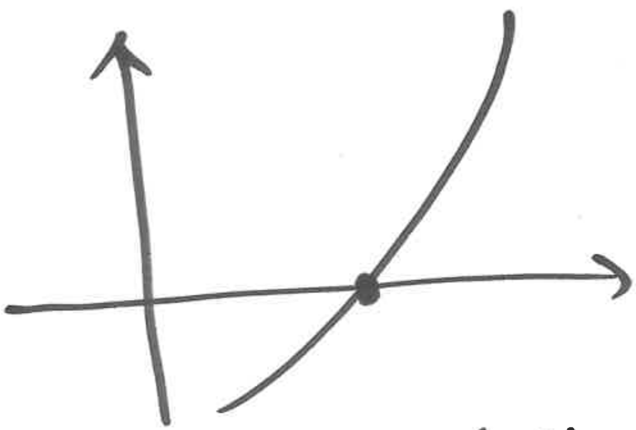
- For smooth functions, this is to say $f^{(k)}(p)=0$, $k=0,1,\cdots,m-1$, $f^{(m)}(p) \neq 0$.

In fact, Taylor expansion at p gives

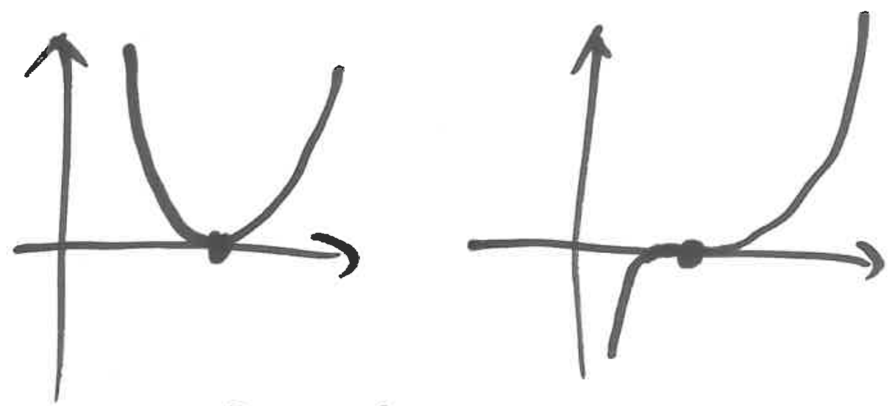
$$f(x) = f(p) + f'(p)\cdot(x-p) + \frac{f''(p)}{2!}(x-p)^2$$

$$+ \cdots + \frac{f^{(m-1)}(p)}{(m-1)!}(x-p)^{m-1} + \underbrace{\frac{f^{(m)}(p)}{m!}(x-p)^m} + O\left((x-p)^{m+1}\right)$$

$$\approx \frac{f^{(m)}(p)}{m!}(x-p)^m$$



$m=1$ (simple root)



$m \geq 2$ (multiple roots)

- For multiple roots, Newton's method only has linear convergence

Say, $f(x) \approx a(x-p)^m$, $a \neq 0$, $m \geq 2$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \approx p_n - \frac{a(p_n - p)^m}{am(p_n - p)^{m-1}}$$

$$= p_n - \frac{1}{m}(p_n - p)$$

$$p_{n+1} - p \approx (p_n - p) - \frac{1}{m}(p_n - p)$$

$$= \underbrace{\left(1 - \frac{1}{m}\right)}_{\text{"}k\text{"} \in (0,1)} (p_n - p)$$

• Modified Newton's method

Define $\mu(x) = \dfrac{f(x)}{f'(x)}$

• Any root p of f becomes a simple root of μ.

To see this, if $f(x) \approx a(x-p)^m$,

then $\mu(x) \approx \dfrac{a(x-p)^m}{am(x-p)^{m-1}} = \frac{1}{m}(x-p)$

$$\mu'(x) = \frac{(f'(x))^2 - f(x)\cdot f''(x)}{(f'(x))^2}$$

Applying Newton's method to $\mu(x)$ gives

$$p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)} = p_n - \frac{f(p_n)\cdot(f'(p_n))^2}{f'(p_n)\left[(f'(p_n))^2 - f(p_n)f''(p_n)\right]}$$

$$= p_n - \frac{f(p_n)\cdot f'(p_n)}{(f'(p_n))^2 - f(p_n)f''(p_n)}$$

- This achieves quadratic convergence for any root (even w/ multiplicity), but much more expensive (need f'')