Ex Use fixed point iteration to find the root of $f(x) = x^3 + 4x^2 - 10$ on $[1,2]$.

(notice that $f'(x) = 3x^2 + 8x \in [11, 28]$ for $x \in [1,2]$)

Choose $g_1(x) = x - f(x)$ $\longrightarrow$ f.p. thm doesn't apply

$g_2(x) = x - \dfrac{f(x)}{20}$ $\longrightarrow$ f.p. thm applies

$g_3(x) = x - \dfrac{f(x)}{100}$

$g_4(x) = x - \dfrac{f(x)}{f'(x)}$

$$g_1'(x) = 1 - f'(x) \in [-27, -10]$$

$$g_2'(x) = 1 - \frac{f'(x)}{20} \in \left[-\frac{8}{20}, \frac{9}{20}\right]$$

$$g_3'(x) = 1 - \frac{f'(x)}{100} \in \left[\frac{72}{100}, \frac{89}{100}\right]$$

- If we don't know k, how to set TOL?

$\longrightarrow$ use $|p_n - p_{n-1}|$ as an error indicator.

for linear convergence,

$$p_n - p \approx Ck^n \quad , \quad p_{n-1} - p \approx Ck^{n-1}$$

$$p_n - p_{n-1} \approx Ck^n - Ck^{n-1} = Ck^n \cdot \left(1 - \frac{1}{k}\right)$$

is also of order $O(k^n)$, same as $|p_n - p|$

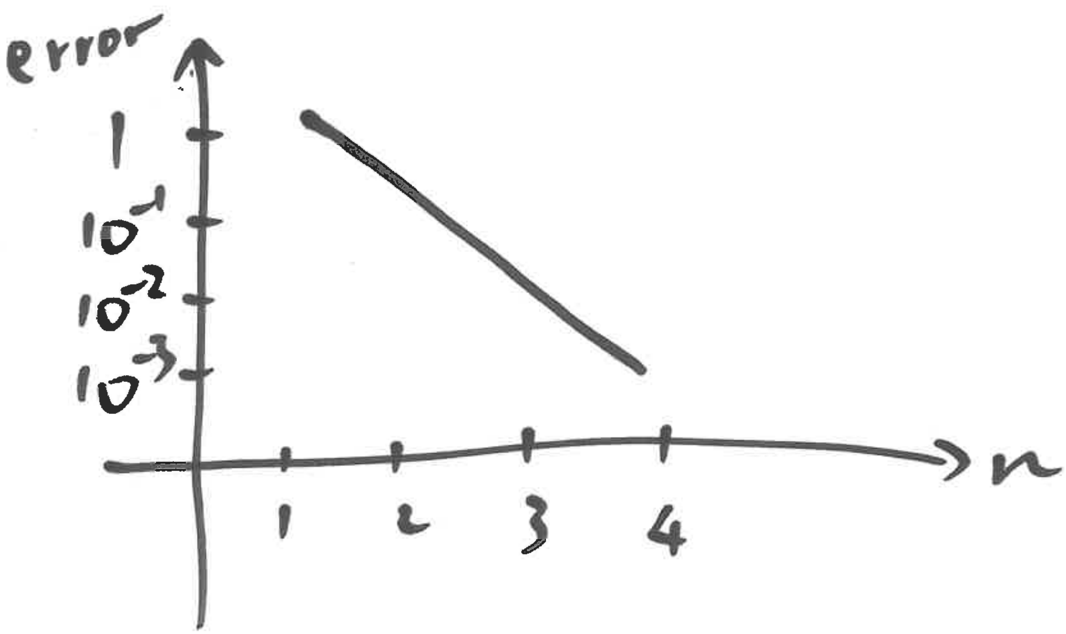
- Set MAX_IT to avoid dead loops.

- How to check linear convergence?

$$|p_n - p| \approx C k^n \qquad 0 < k < 1$$

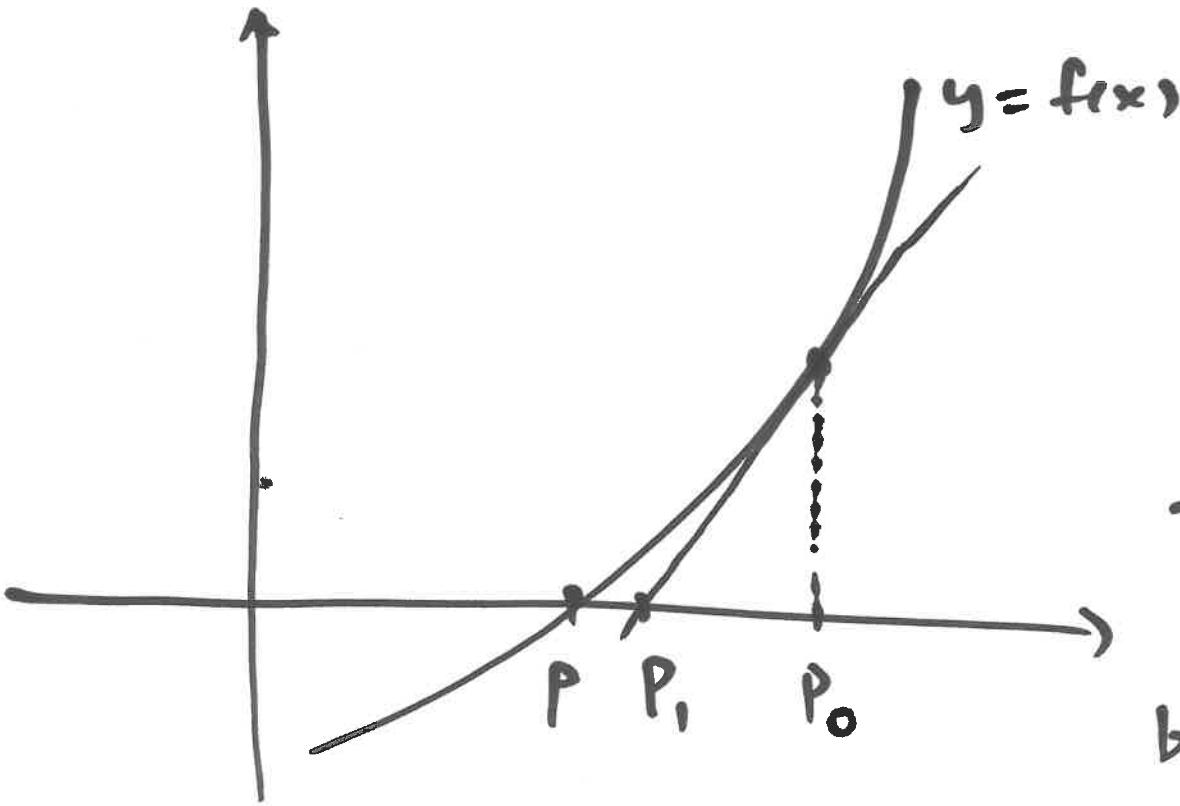
$$\ln |p_n - p| \approx \ln C + n \ln k$$

$\Rightarrow$ $\ln |p_n - p|$ is a linear function of n



2.3 Newton's method

Find the root p of $f(x)$



If $|p_0 - p|$ is quite small, then $f(x)$ on $[p, p_0]$ is well-approximated by the tangent line at p_0.

$$0 = f(p) \approx f(p_0) + f'(p_0)(p - p_0)$$

Set RHS = 0 gives

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

<u>Newton's method</u>

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad n \geq 1$$

- One expects Newton's method to be good only when $|p_0 - p|$ is small
- Newton's method is a special case of f. p. i.

$$p_n = g(p_{n-1}) \quad , \quad g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{(f'(x))^2}$$

$$g'(p) = 1 - \frac{(f'(p))^2 \cancel{- f(p) \cdot f''(p)}}{(f'(p))^2} = 0$$

__Thm__ Let $g \in C^2$. Suppose p is a fixed point of g w/ $g'(p) = 0$. Then $\exists\, \delta > 0$ s.t. f.p.i. starting inside $[p-\delta, p+\delta]$ converges to p, w/ error estimate

$$|p_{n+1} - p| \le C\,|p_n - p|^2, \quad \forall\, n \ge 0$$

"quadratic convergence"

C^k : functions w/ k-th order continuous derivatives

- Why convergence?

$g'(p) = 0 \rightsquigarrow |g'(x)|$ is small when

x is close to p.

In particular, $|g'(x)| \le \frac{1}{2}$

$\rightsquigarrow$ apply f. p. thm.

- Why quadratic convergence?

$$p_{n+1} - p = g(p_n) - g(p)$$

$$\approx \left(\cancel{g(p)} + \underbrace{\cancel{g'(p)\cdot(p_n - p)}}_{=0} + \frac{g''(p)}{2}(p_n - p)^2 \right) \cancel{- g(p)}$$

$$= \frac{g''(p)}{2}(p_n - p)^2$$

For general f. p. i.

Adventages:

- can be generalized to multi-D.
- only need $g(x)$ (not $g'(x), \cdots$)

Disadvantage:

- hard to check assumption (cond. stability)

For Newton's method:

Advantage: fast convergence.

Disadvantages:

- cond. stability.
- need $f'(x)$ (this can be expensive in multi-D)

$$\vec{f}(\vec{x}) = \begin{pmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_n \end{pmatrix} \qquad \nabla \vec{f}(x) = \Big(\qquad \Big)_{n \times n}$$