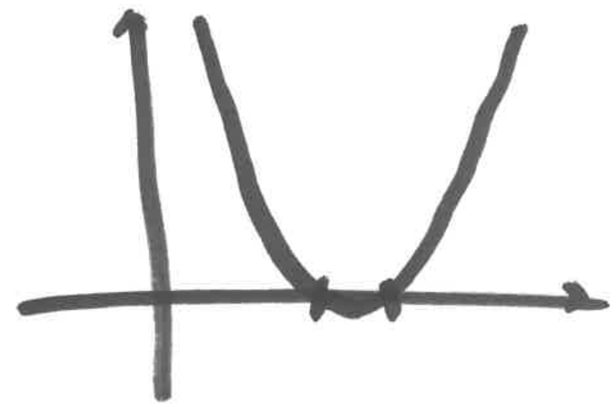
Bisection method

Advantages:

- Doesn't require too many properties of $f(x)$ (just continuous)
- Only needs values of $f(x)$ (no $f'(x), \dots$)

Disadvantages:

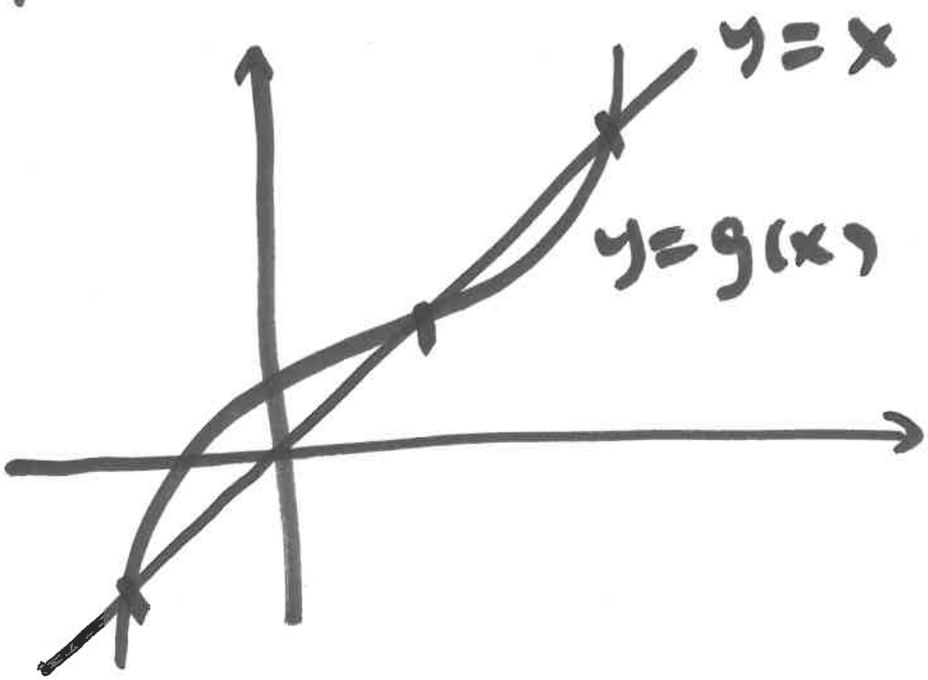
- Only works for 1D, real numbers
- Sometimes it's hard to find a, b s.t. $f(a)f(b) < 0$



- Can only find one of the roots.

2.2 Fixed point iteration

Def p is a **fixed point** of a function $g(x)$ if $g(p) = p$



- Finding roots is equivalent to finding fixed points

$$f(p) = 0 \iff g(p) = p - f(p) = p$$

$$\iff \tilde{g}(p) = p - \frac{f(p)}{100} = p$$

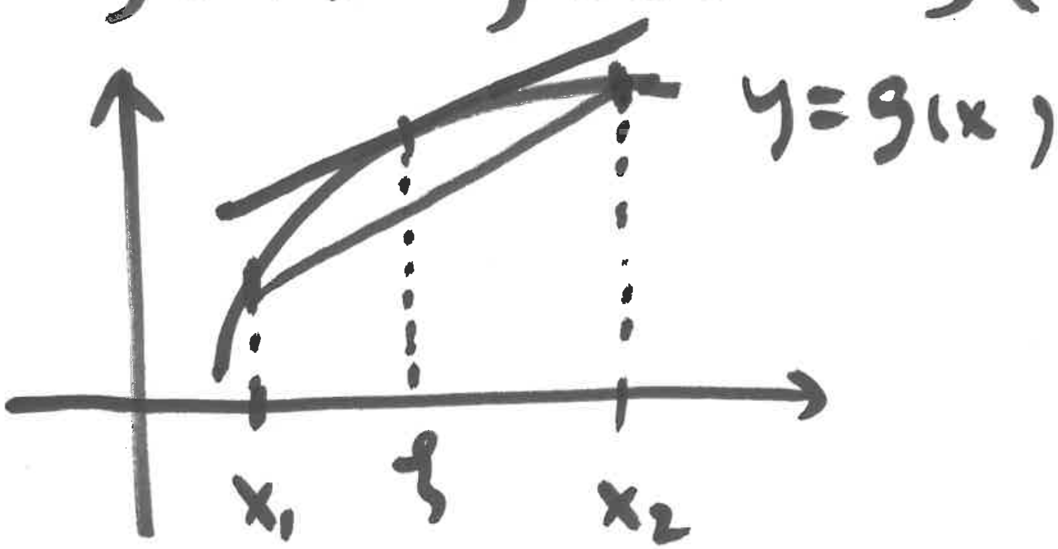
(many choices of g)

Mean value theorem

Let $g(x)$ be differentiable on (a,b)

For any $x_1, x_2 \in (a,b)$, $\exists\, \xi$ between x_1 and x_2 s.t.

$$g(x_1) - g(x_2) = g'(\xi)(x_1 - x_2)$$



Thm (Fixed point theorem / contraction mapping theorem)

Let $g \in C[a,b]$ s.t.

$g(x) \in [a,b]$, $\forall x \in [a,b]$

Assume $g'(x)$ exists on (a,b) and

(*) $|g'(x)| \leq k, \forall x \in (a,b)$ for some $0 < k < 1$.

Then,

(1) there <u>exists</u> a <u>unique</u> fixed point p of g on $[a,b]$.

(2) For any $p_0 \in [a,b]$, the sequence $\{p_n\}$ defined by

$$p_n = g(p_{n-1}),\ \forall n \geq 1$$

converges to p.

"fixed point iteration"

- In fact, for $(*)$, it suffices to require a weaker condition

$$(*)' \quad |g(x_1)-g(x_2)| \le k|x_1-x_2|$$

$$\forall x_1, x_2 \in [a,b] \quad \text{for some } 0<k<1$$

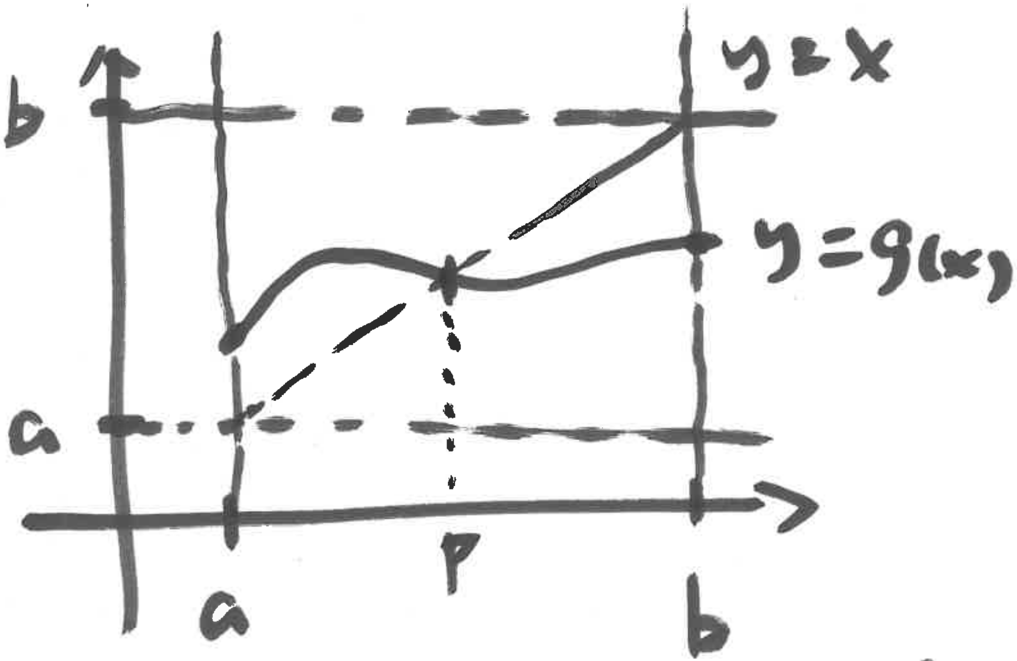
To see $(*) \Rightarrow (*)'$.

$$|g(x_1)-g(x_2)| = |g'(\xi)(x_1-x_2)|$$

(by m.v.t.)

$$\le k \cdot |x_1-x_2| \quad (\text{by } (*))$$

Pf) ① Existence of p.



Define ~~$h(x)=x-g(x)$~~

$h(x)=g(x)-x$

$$h(a)=g(a)-a$$
$$\geq a-a=0$$
$$h(b)=g(b)-b$$
$$\leq b-b=0$$

By intermediate value theorem,

$\exists\, p \in [a,b]$ s.t. $h(p)=0 \Rightarrow g(p)=p$

② Uniqueness of p.

Suppose $p_1, p_2 \in [a,b]$ are fixed points of g. Then

$$|p_1 - p_2| = |g(p_1) - g(p_2)| \quad \text{(fixed pt)}$$

$$\le k\,|p_1 - p_2| \quad \text{(by } (*)'\text{)}$$

$$\Rightarrow \quad |p_1 - p_2| = 0 \quad (\text{by } 0 < \underline{k < 1})$$

strict less !

③ $p_n \longrightarrow p$

$$|p_n - p| = |\underset{\substack{\uparrow \\ (\text{f.p.i.})}}{g(p_{n-1})} - \underset{\substack{\uparrow \\ (p \text{ is f.p.})}}{g(p)}|$$

$$\leq k\,|p_{n-1} - p| \quad (\text{by } (*)')$$

$$\leq k^2\,|p_{n-2} - p|$$

$$\leq \cdots \leq k^n\,|p_0 - p| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$(\text{by } 0 < \underline{k < 1})$$

Fixed point iteration

To find the fixed point p of g, start w/ p_0, compute

$$p_n = g(p_{n-1}), \ \forall n \geq 1$$

...

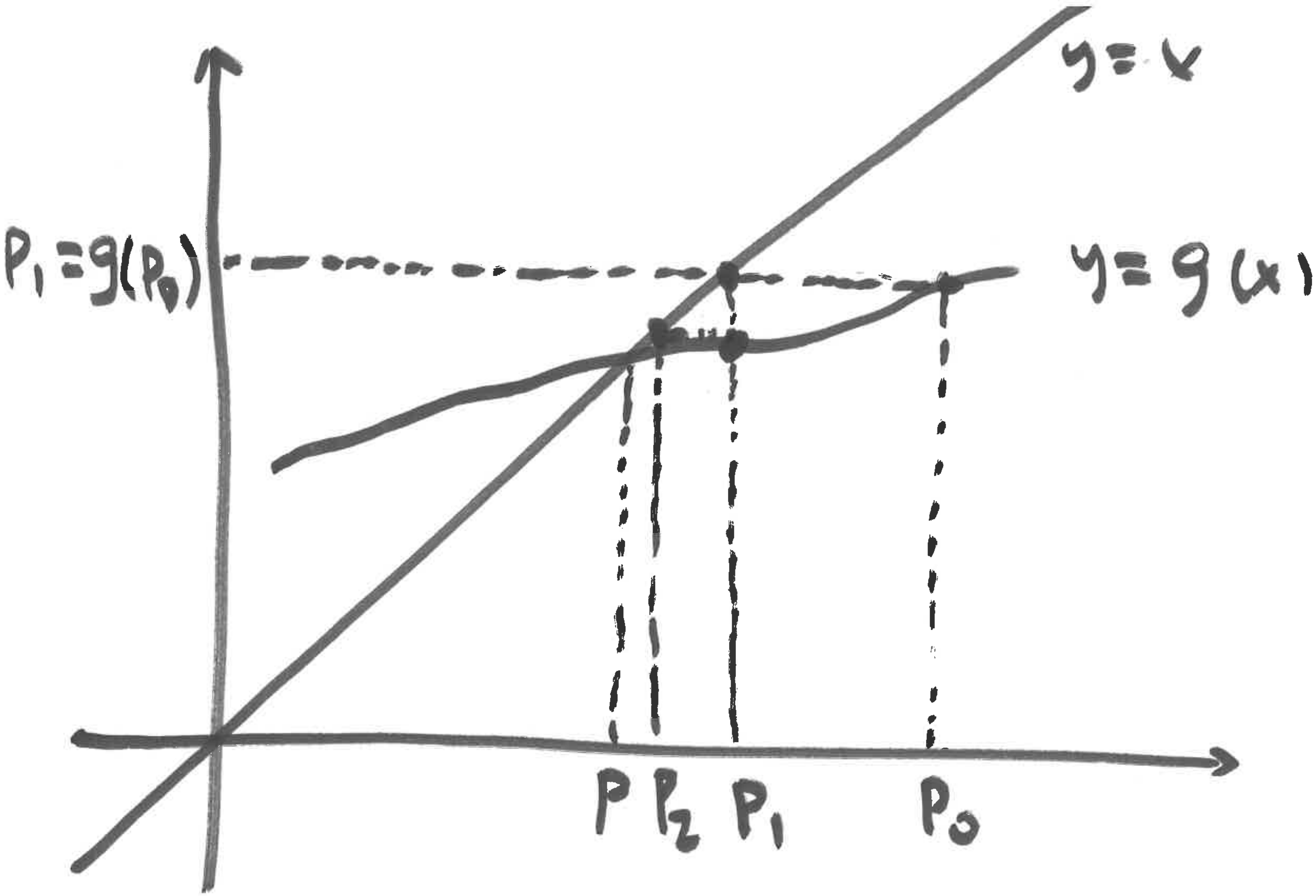
If the assumptions of f. p. thm. are satisfied, then it is guaranteed $p_n \to p$, w/ error estimate

$$|p_n - p| \leq (b-a) \cdot k^n$$

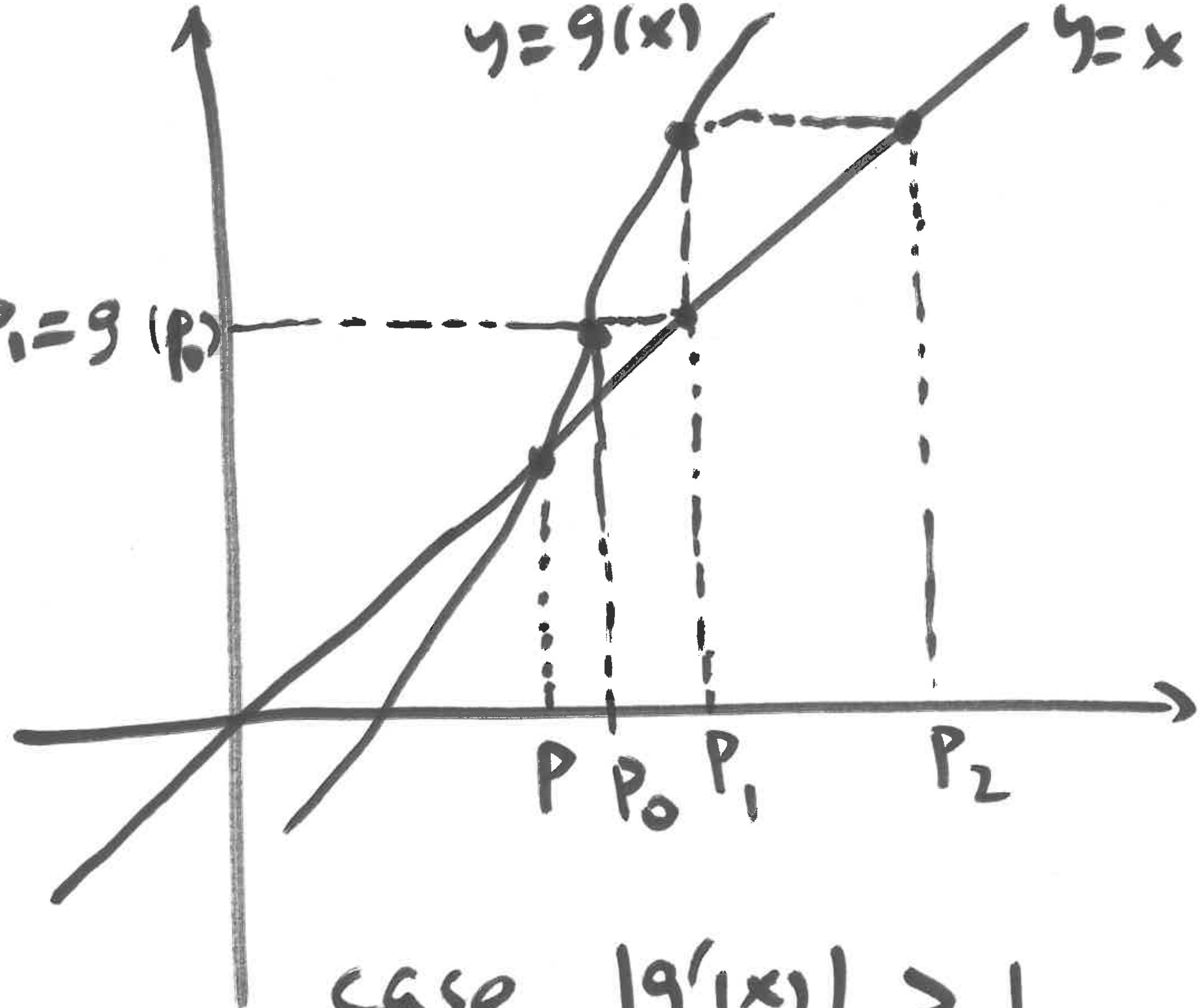
rate of convergence is $O(k^n)$

"linear convergence".

- f. p. i. is conditionally stable, because the assumptions of f. p. thm. are hard to check.
- f. p. i. can be generalized to higher dimensions ($\mathbb{R}^n$, ...)



case $|g'(x)| \le k < 1$, $p_n \to p$



case $|g'(x)| > 1$

$p_n \not\to p$