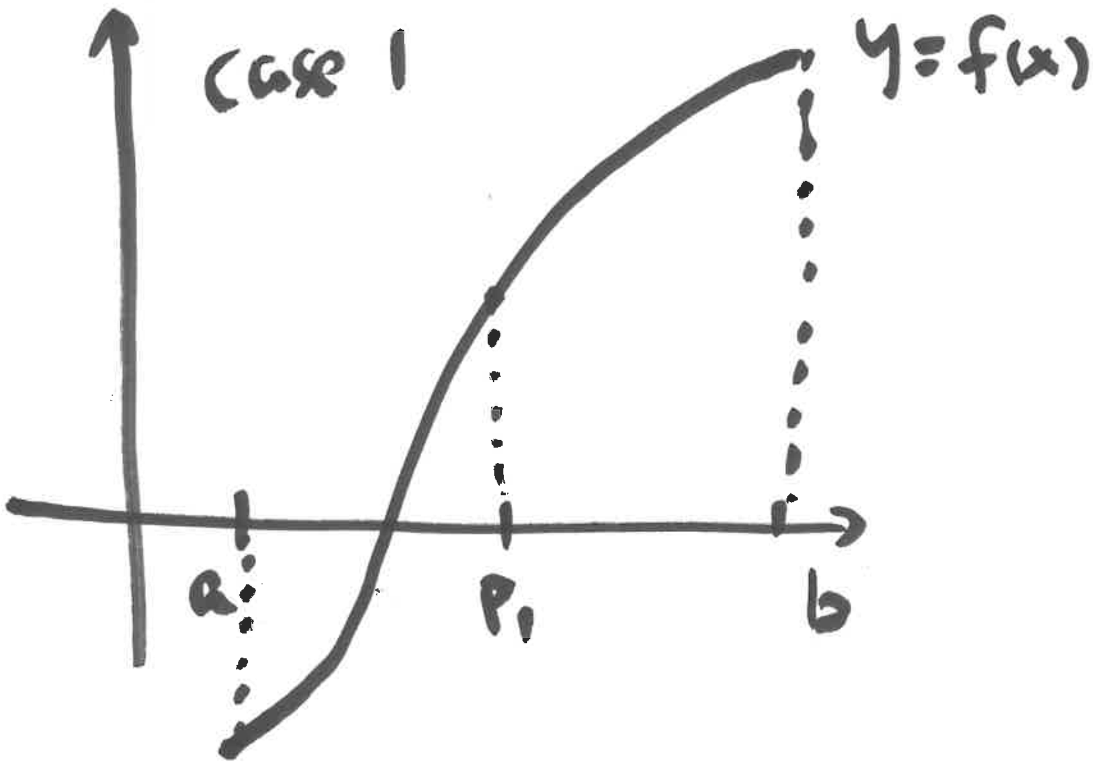
2.1 Bisection method

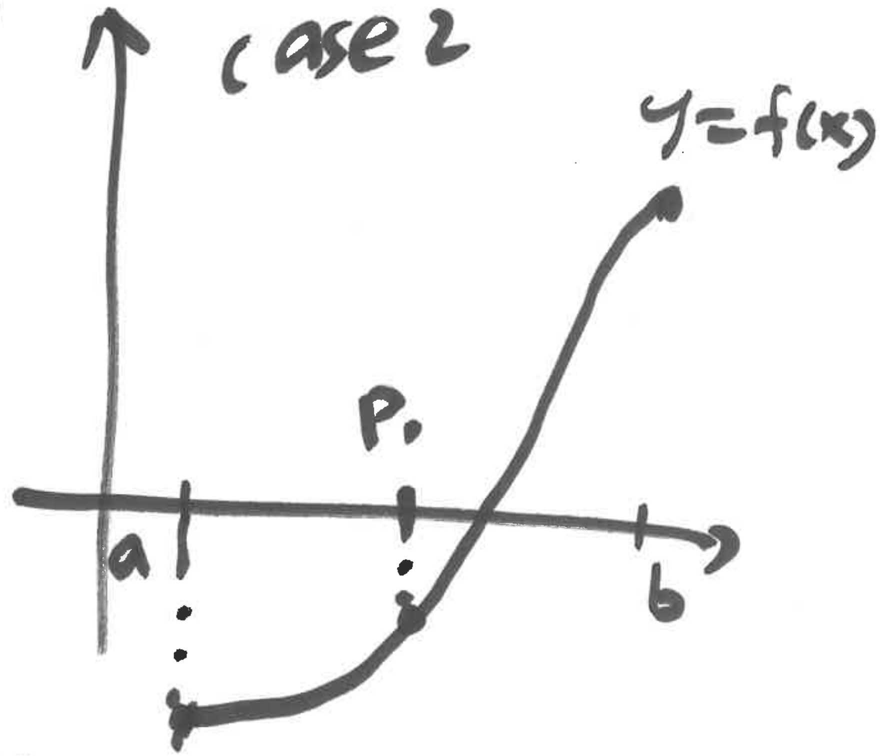
Question: Given $f(x)$, find x

s.t. $f(x) = 0$

Assume f is continuous on $[a, b]$

Assume $f(a)\,f(b) < 0$





Intermediate value theorem:

$\exists\ p \in (a,b)$ s.t. $f(p)=0$

Bisection method:

Take $p_1 = \dfrac{a+b}{2}$

Is $f(a)f(p_1) > 0$, < 0, $= 0$?

- If $f(a)f(p_1) < 0$, then $\exists$ root in (a, p_1)

$\rightsquigarrow$ replace $[a,b]$ by $[a, p_1]$

- If $f(a)f(p_1) > 0$, then $f(p_1)f(b) < 0$, $\exists$ root in (p_1, b)

$\rightsquigarrow$ replace $[a,b]$ by $[p_1, b]$.

- If $f(a)f(p_1) = 0$, then $f(p_1) = 0$

  $\rightsquigarrow$ Output p_1

stop if $b-a$ is sufficiently small

(set TOL)

<u>Thm</u> (error estimate of bisection)

Suppose $f \in C[a,b]$, $f(a)f(b) < 0$.

$\uparrow$

continuous functions on $[a,b]$

Then $\{p_n\}$ generated by bisection method approximates some root p of f, w/

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \forall n \ge 1.$$

pf) $a_1 = a$, $b_1 = b$

$$b_2 - a_2 = \frac{1}{2}(b-a)$$

$$b_3 - a_3 = \frac{1}{2^2}(b-a)$$

$$\vdots$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a)$$

$$|p - p_n| \le \frac{1}{2}(b_n - a_n)$$

Ex Approximate $\sqrt{2}$ by applying bisection to $f(x) = x^2 - 2$, 4 times, starting from $[1, 2]$.

Check: $f(1) \cdot f(2) = (-1) \cdot 2 < 0$

$[a_1, b_1] = [1, 2]$, $p_1 = 1.5$

$\underline{f(a_1) = -1}$, $f(b_1) = 2$, $\underline{f(p_1) = 0.25}$

$[a_2, b_2] = [1, 1.5]$, $p_2 = 1.25$

$f(a_2) = -1$, $\underline{f(b_2) = 0.25}$, $\underline{f(p_2) = -0.4375}$

$[a_3, b_3] = [1.25, 1.5]$, $p_3 = 1.375$

$f(a_3) = -0.4375$, $\underline{f(b_3) = 0.25}$, $\underline{f(p_3) = -0.10\ldots}$

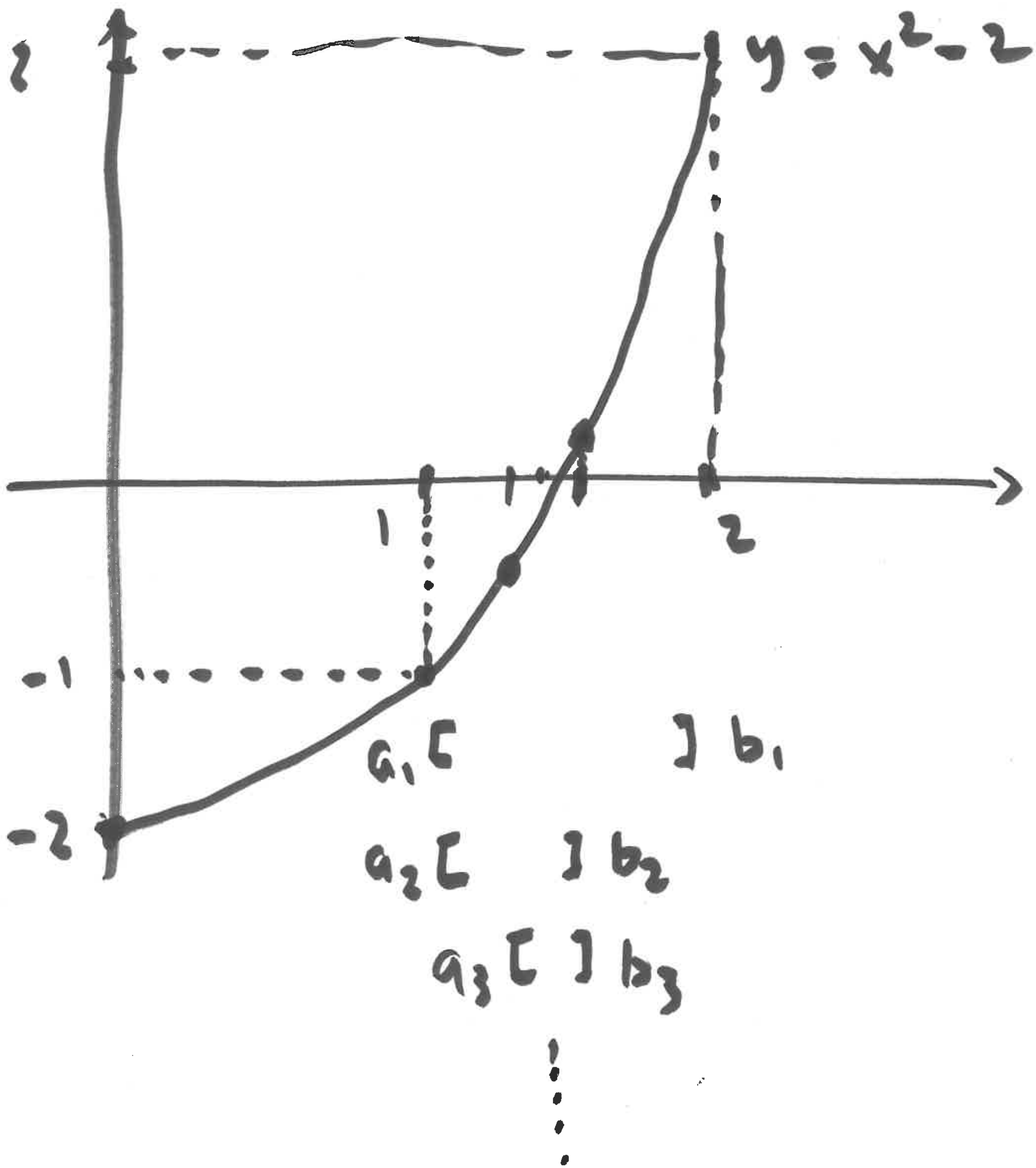
$[a_4, b_4] = [1.375, 1.5]$, $\boxed{p_4 = 1.4375}$

error estimate:

$$|p_4 - \sqrt{2}| \leq \frac{b-a}{2^4} = \frac{1}{16} = 0.0625$$

true error:

$$|p_4 - \sqrt{2}| = |1.4375 - 1.4142\ldots|$$
$$\approx 0.023$$



$y = x^2 - 2$
2
1
2
-1
-2
a_1 [ ] b_1
a_2 [ ] b_2
a_3 [ ] b_3

How to evaluate a numerical method?

- Accuracy: how large is the error?

  ($|p_n - p| \leq ?$)

- Computational cost:

  how much computation does it take?

  (how many times of $f(x)$ to compute?)

- Stability (robustness):

  does the method work for any input

  w/ round-off errors?

[unconditionally stable : work for any input

Conditionally stable : work for some input

- Bisection method: unc. stable.

Rate of Convergence

Suppose an algorithm generates $\{p_n\}$ to approximate p.

when $\exists\ C > 0$ and a sequence $\{\beta_n\}$, $\beta_n \to 0$, s.t.

$$|p_n - p| \leq C\beta_n \quad , \forall n \geq 1$$

then the algorithm has <u>rate of convergence</u> $O(\beta_n)$

For bisection, $|p_n - p| \le \underbrace{(b-a)}_{C} \cdot \underbrace{\frac{1}{2^n}}_{\beta_n}$

the rate of convergence is $O\left(\frac{1}{2^n}\right)$

"linear convergence"

(defined as r. o. c. of $O(k^n)$, $0 < k < 1$)