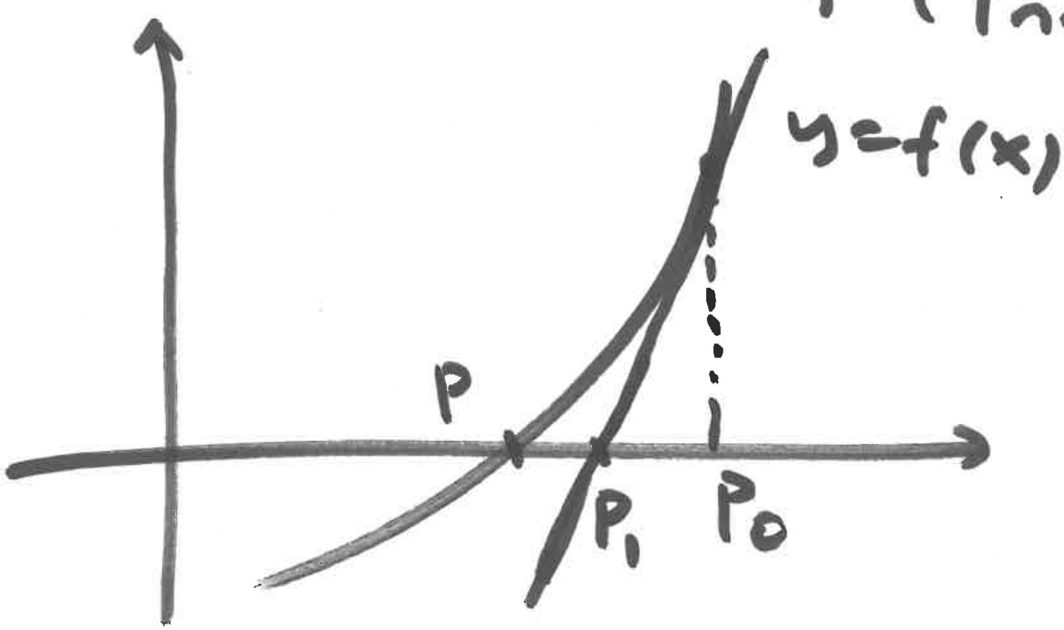
2.3, 2.4 Generalizations of Newton's method

Recall Newton's method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$



Question: how to avoid calculating f'?

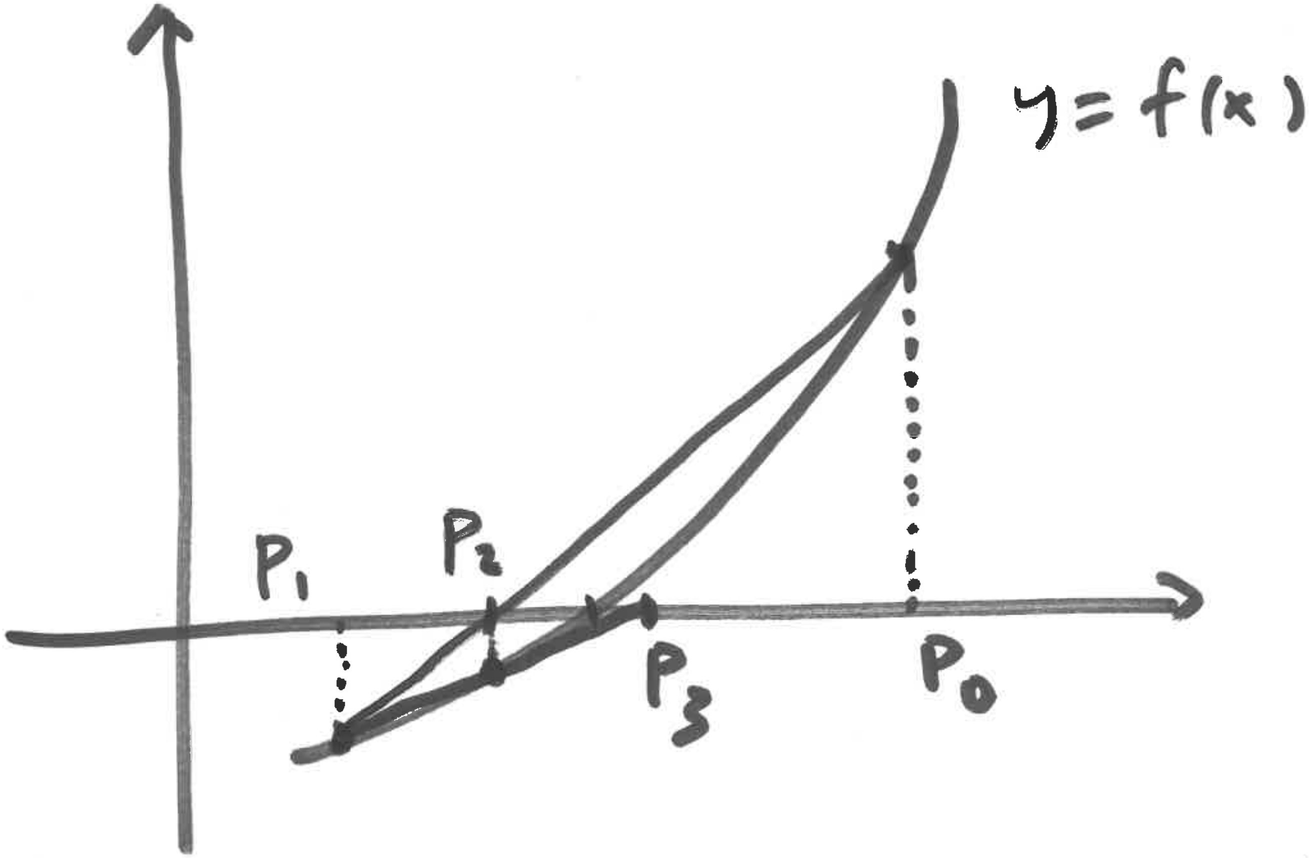
$$\rightsquigarrow \quad f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(\tilde{p})}{p_{n-1} - \tilde{p}}$$

for some $\tilde{p}$ near p_{n-1}.

- <u>The secant method</u>: take $\tilde{p} = p_{n-2}$

$$p_n = p_{n-1} - \frac{f(p_{n-1}) \cdot (p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

$n \geq 2$



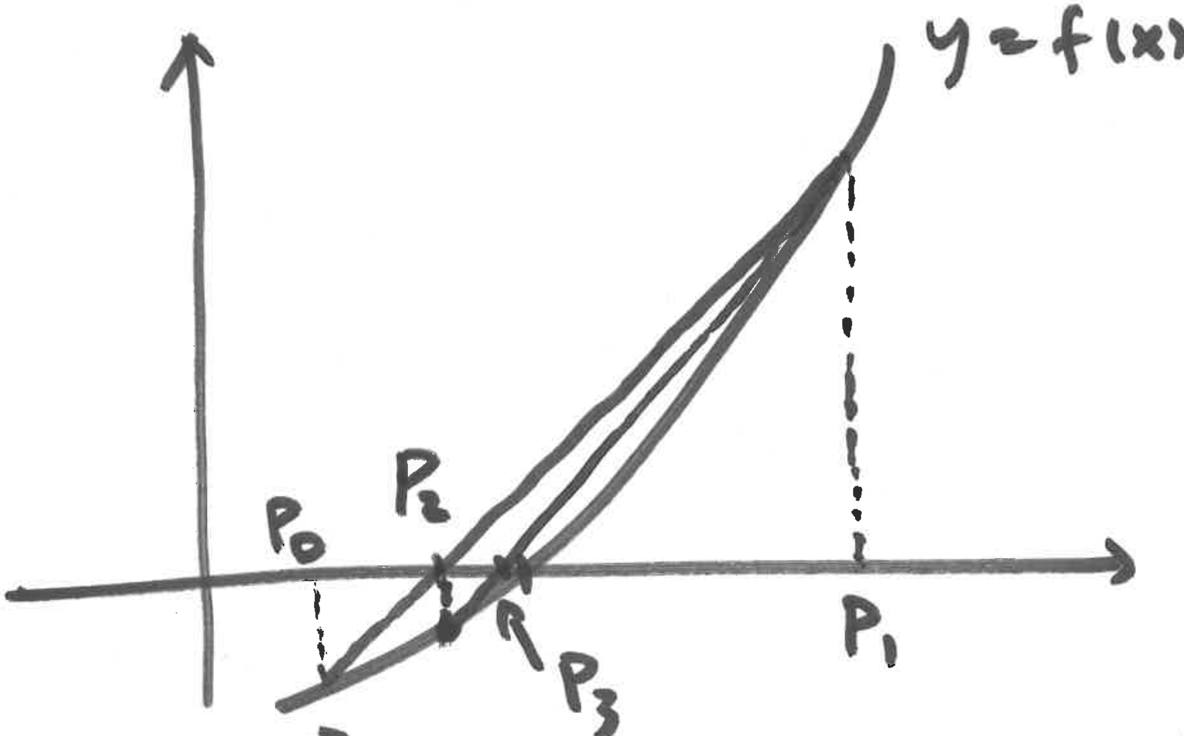
- The convergence rate is faster than linear, slower than quadratic.

$$|p_{n+1} - p_n| \le C \, |p_n - p_{n-1}|^{\alpha}$$

$$\alpha = \frac{\sqrt{5}+1}{2} \approx 1.6\cdots$$

- **The method of false position:**

combine secant method w/ bisection.



Starting from $[p_0, p_1]$ w/ $f(p_0) \cdot f(p_1) < 0$

Compute $p = p_1 - \dfrac{f(p_1) \cdot (p_1 - p_0)}{f(p_1) - f(p_0)}$

If $|p - p_1| < TOL$, break

If $f(p) \cdot f(p_1) < 0$

replace p_0, p_1 by p_1, p

else

replace p_0, p_1 by p_0, p

Output p (as approximation).

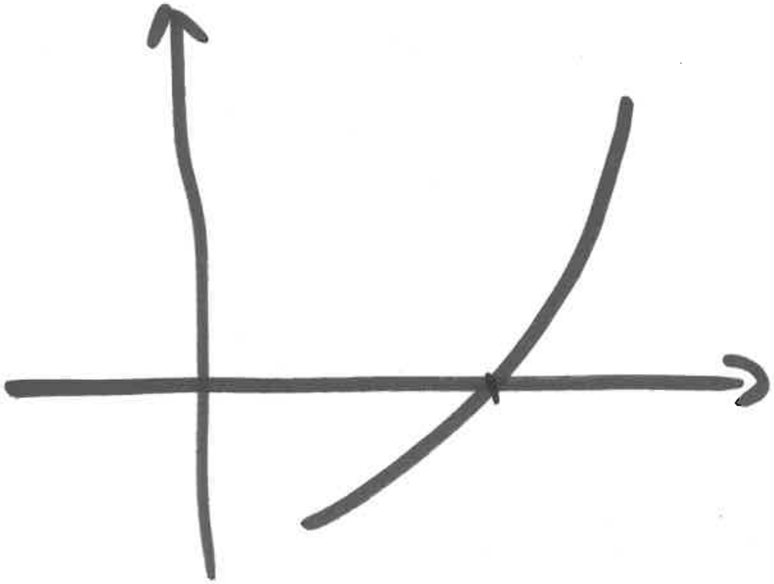
- Generally, convergence rate is linear.
- Can guarantee convergence (like bisec.) the final interval is not small.

<u>Question</u>: What happens when applying Newton's to multiple roots?

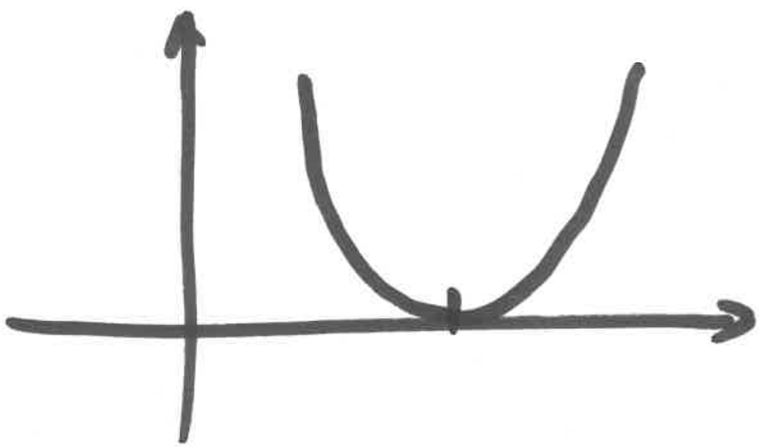
<u>Def</u> A solution p of $f(x)=0$ is called a <u>root (zero) of multiplicity m</u> if $\exists\, q(x)$ s.t. $f(x) = (x-p)^m q(x)$, $\lim_{x \to p} q(x) \neq 0$.

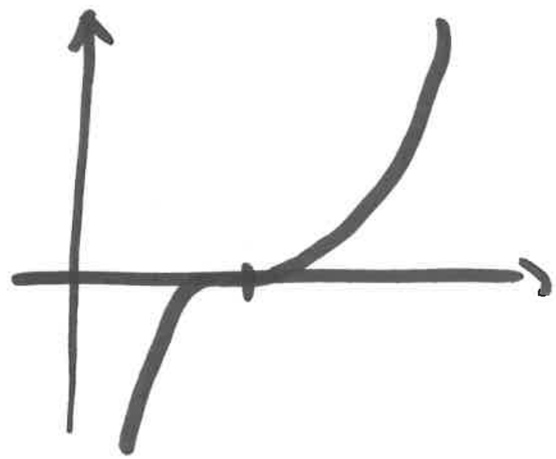
- For smooth functions, this is to say

$$f^{(k)}(p) = 0,\ k = 0, \cdots, m-1,\quad f^{(m)}(p) \neq 0$$



$m=1$ (simple root)





$m \geq 2$ (multiple root)

- For multiple roots, Newton's method only has linear convergence.

Say $f(x) \approx a(x-p)^m$ near p

$m \geq 2 \qquad a \neq 0$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} \approx p_n - \frac{a(p_n-p)^m}{a \cdot m(p_n-p)^{m-1}}$$

$$= p_n - \frac{1}{m}(p_n - p)$$

$$p_{n+1} - p \approx (p_n - p) - \frac{1}{m}(p_n - p) = \left(1 - \frac{1}{m}\right)(p_n - p)$$

$$0 < 1 - \frac{1}{m} < 1$$

$$|p_n - p| \approx O\left(\left(1 - \frac{1}{m}\right)^n\right)$$

- <u>Modified Newton's method</u>

Define $\mu(x) = \dfrac{f(x)}{f'(x)}$

- Any root p of f becomes a simple root of μ.

Pf) If $f(x) = (x-p)^m q(x)$, $q(p) \neq 0$

$$\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$

$$= \boxed{\frac{q(x)}{m\,q(x) + (x-p)q'(x)}}(x-p).$$

$= \phi(x)$

$$\phi(p) = \frac{q(p)}{m\,q(p) + \cancel{(p-p)q'(p)}} = \frac{1}{m} \neq 0.$$

- Apply Newton's to $\mu(x)$:

$$P_{n+1} = g(P_n)$$

$$g(x) = x - \frac{\mu(x)}{\mu'(x)} = x - \frac{f(x)\cdot f'(x)}{(f'(x))^2 - f(x)f''(x)}$$

- Need f''
- For simple roots, it reduces to Newton's
- quadratic convergence for any root.