Bisection:

Advantages:

- Doesn't require too many properties of the function f.
- Only needs values of $f(x)$ (no $f'(x), \dots$)

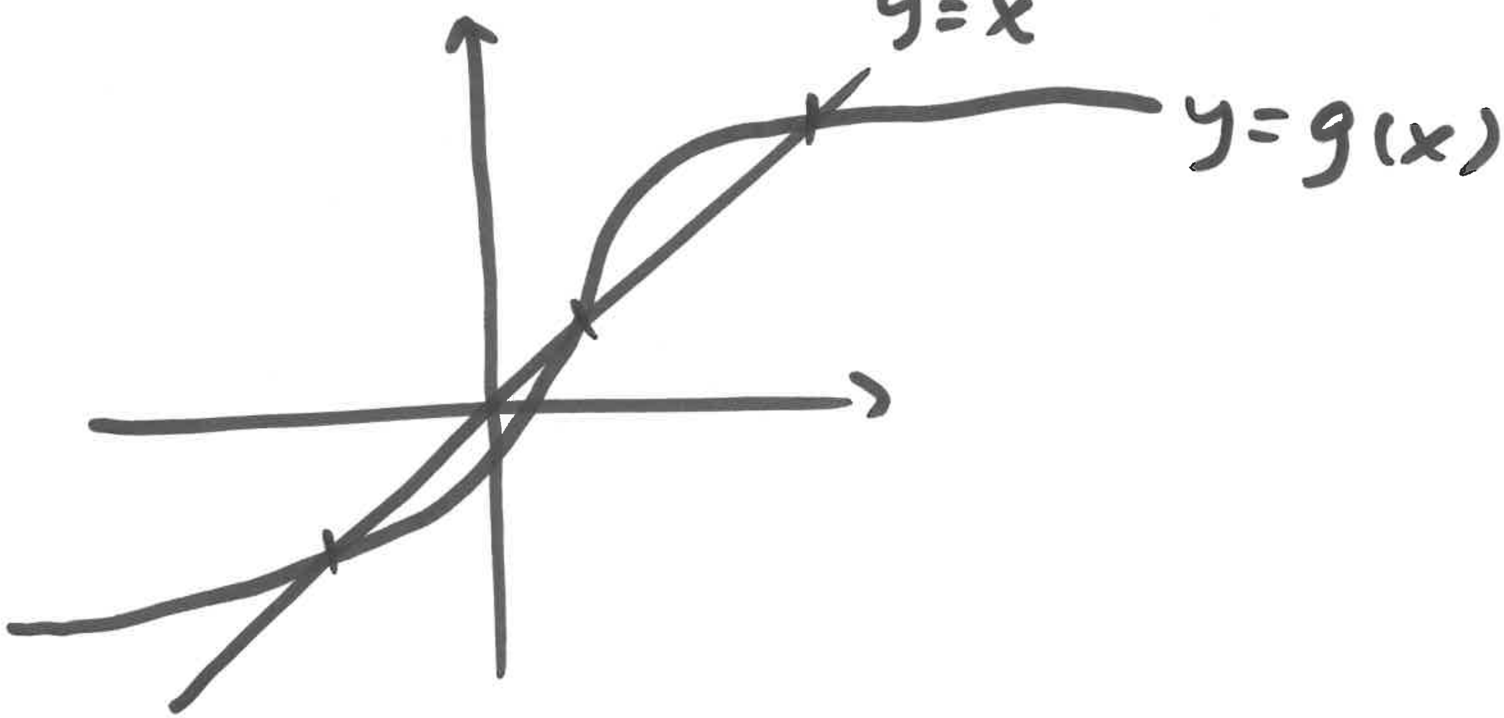
Disadvantages:

- Only works for 1D, real numbers
- Sometimes it's hard to find a, b s.t. $f(a) f(b) < 0$

- Can only find one of the roots

2.2 Fixed point iteration

Def p is a fixed point of a function g if $g(p) = p$



- Finding roots is equivalent to finding fixed points

$$f(p) = 0 \iff g(p) := p - f(p) = p$$

$$\tilde{g}(p) := p - \frac{f(p)}{100} = p$$

Thm (Fixed point theorem / Contraction mapping theorem)

Let $g \in C[a,b]$ s.t. $g(x) \in [a,b]$ $\forall x \in [a,b]$

Assume $\exists$ $0 < \underline{k < 1}$ s.t.

$$(*) \quad |g(x_1) - g(x_2)| \leq k\,|x_1 - x_2| \quad \forall x_1, x_2 \in [a,b].$$

"Contraction"

Then there <u>exists unique</u> fixed point p of g in $[a,b]$.

Also, for any $p_0 \in [a, b]$, the sequence $\{p_n\}$ defined by

$$p_n = g(p_{n-1}) \quad \forall n \geq 1$$

converges to p.

- The condition $|g'(x)| \leq k$ implies (*). However, (*) is more general.
- This theorem works for more general spaces (for example, $\mathbb{R}^n$, ...)

pf) ① Uniqueness

Suppose p and $\tilde{p}$ are fixed points of g.

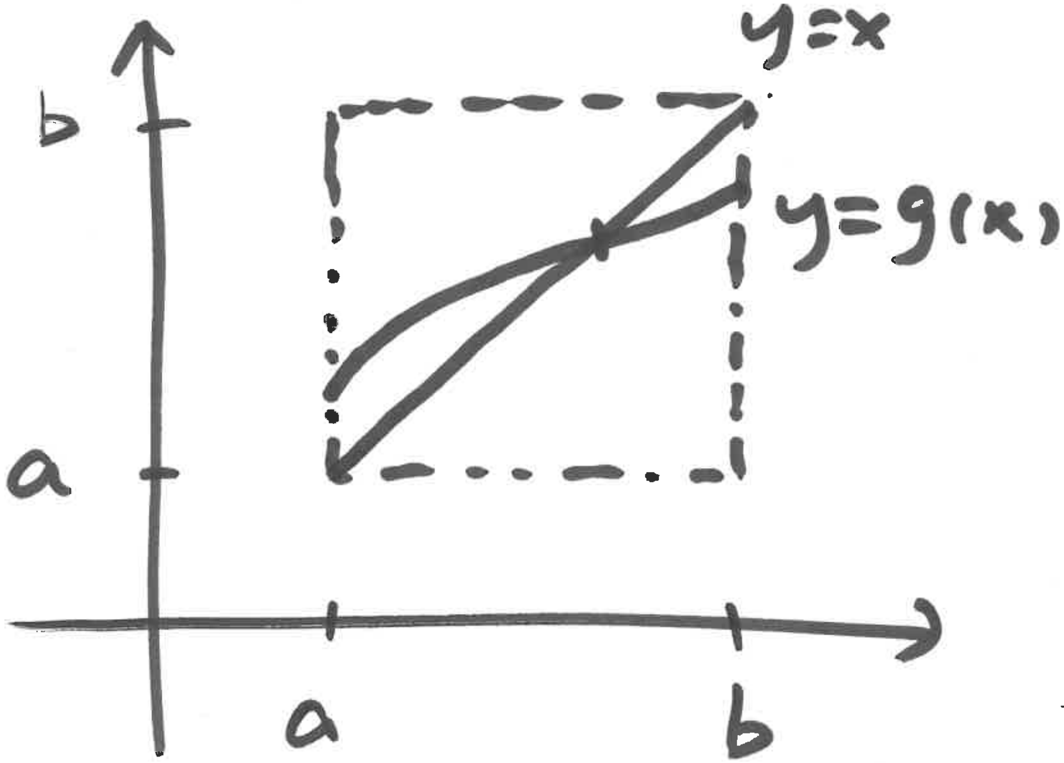
$$|p-\tilde{p}| = |g(p)-g(\tilde{p})|$$

$$\leq k|p-\tilde{p}| \quad \text{by } (*)$$

$k<1$

$$\Rightarrow p = \tilde{p}$$

② Existence, $p_n \to p$



$h(x) = x - g(x)$

continuous

$h(a) = a - g(a) \le 0$

$h(b) = b - g(b) \ge 0$

$\Longrightarrow$ $h(p) = 0$

intermediate value thm

for some p.

To show $p_n \to p$

$$|p_n - p| \le |g(p_{n-1}) - g(p)|$$

$$\le k\,|p_{n-1} - p| \quad \text{by } (*)$$

$$\le \cdots \le k^n\,|p_0 - p| \to 0 \text{ as } n \to \infty$$

$\underline{k < 1}$

Method 2: use Cauchy's criterion

$$|p_{n+1} - p_n| \leq |g(p_n) - g(p_{n-1})|$$

$$\leq k\,|p_n - p_{n-1}| \quad \text{by } (*)$$

$$\leq \cdots \leq k^n\,|p_1 - p_0|$$

$$|p_{n+m} - p_n| \leq \sum_{i=0}^{m-1} |p_{n+i+1} - p_{n+i}|$$

$$\leq \sum_{i=0}^{m-1} k^{n+i} \cdot |p_1 - p_0|$$

$$\leq k^n \cdot \frac{1-k^m}{1-k} \cdot |p_1 - p_0| \leq \frac{|p_1 - p_0|}{1-k} \cdot k^n \to 0$$

as $n \to \infty$

$\Rightarrow$ $\{p_n\}$ converges, say, to p

To show p is a fixed point of g,

$$p_n = g(p_{n-1})$$

$$\Big\downarrow \, n \to \infty$$

$$p = g(p)$$

Fixed point iteration:

To find fixed point p of g,
start with some initial value p_0,
compute $p_n = g(p_{n-1})$, $\forall\, n \geq 1$

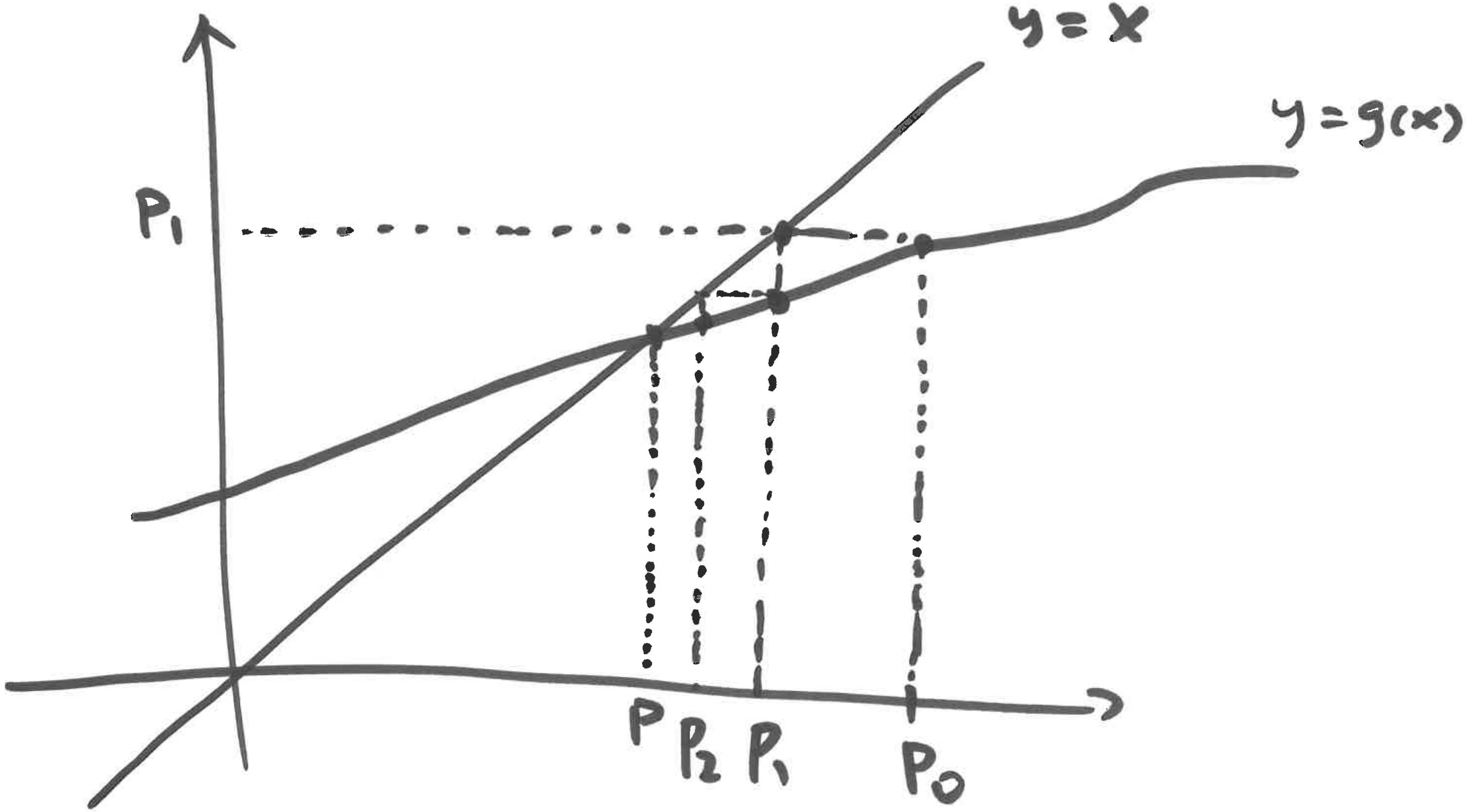
The fixed point theorem guarantees
that under the assumptions,
$p_n \to p$ as $n \to \infty$

Error estimate:

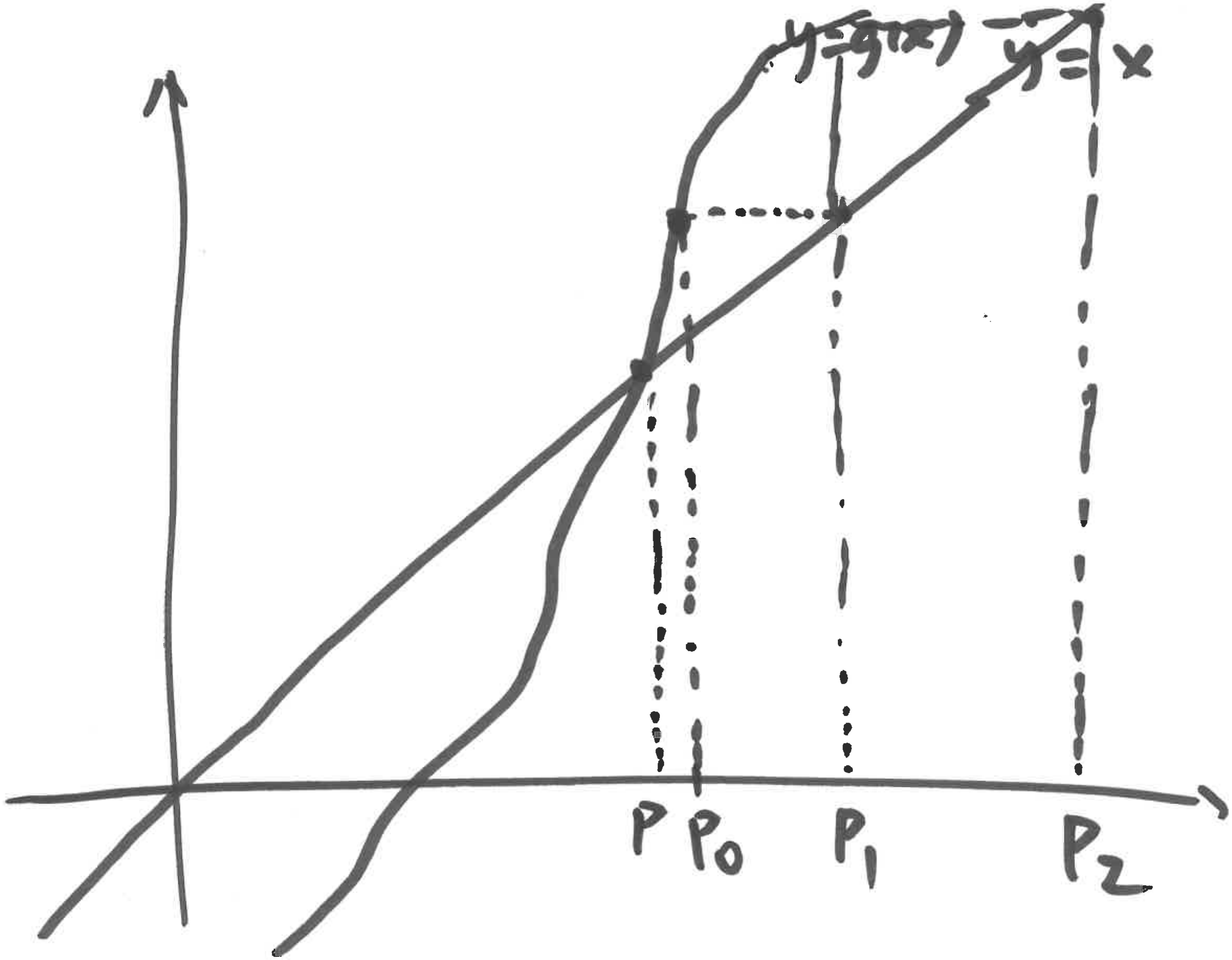
$$|p_n - p| \le k^n (b-a)$$

rate of convergence is $O(k^n)$

"linear convergence"



fixed point iteration works



fixed point iteration doesn't work

$g'(x) > 1$