2.1 Bisection method

1.3 Stability, convergence

Given $f(x)$, find x s.t. $f(x) = 0$

Assume f is continuous.

Suppose for $[a, b]$, there holds

$$f(a)f(b) < 0$$

Intermediate value theorem:

$\exists\, p \in (a,b)$ s.t. $f(p) = 0$

Start from $[a,b]$ with $f(a)\,f(b) < 0$

$$\rightarrow p = a + \frac{b-a}{2}$$

Is $f(a)\,f(p) > 0$ or < 0 or $= 0$?

If $f(a)\,f(p) < 0$ there has to be a root in (a,p)

$\rightsquigarrow$ replace $[a,b]$ by $[a,p]$

If $f(a)f(p) > 0 \Rightarrow f(b)f(p) < 0$

$\rightsquigarrow$ replace $[a,b]$ by $[p,b]$

If $f(a)f(p) = 0$ output p

stop if $b-a$ is sufficiently small

<u>Thm</u> Suppose $f \in C[a,b]$, $f(a)f(b)<0$

Then $\{p_n\}$ generated by bisection approximates some root p of f, with $|p_n - p| \le \frac{b-a}{2^n}$ $\forall n \ge 1$

$a_1 = a \qquad b_1 = b$

$$b_2 - a_2 = \frac{1}{2}(b-a)$$

$$b_3 - a_3 = \frac{1}{4}(b-a)$$

$$\vdots$$

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a)$$

$p \in (a_n, b_n) \qquad p_n = a_n + \dfrac{b_n - a_n}{2}$

<u>Ex</u> Approximate $\sqrt{2}$ by applying bisection to $f(x) = x^2 - 2$, 4 times, starting from $[1, 2]$.

$[a_1, b_1] = [1, 2]$ $\quad p_1 = 1.5$

$f(a_1) = -1$ $\quad f(b_1) = 2$ $\quad f(p_1) = 0.25$

$[a_2, b_2] = [1, 1.5]$ $\quad p_2 = 1.25$

$f(a_2) = -1$ $\quad f(b_2) = 0.25$ $\quad f(p_2) = -0.4375$

$[a_3, b_3] = [1.25, 1.5] \quad p_3 = 1.375$

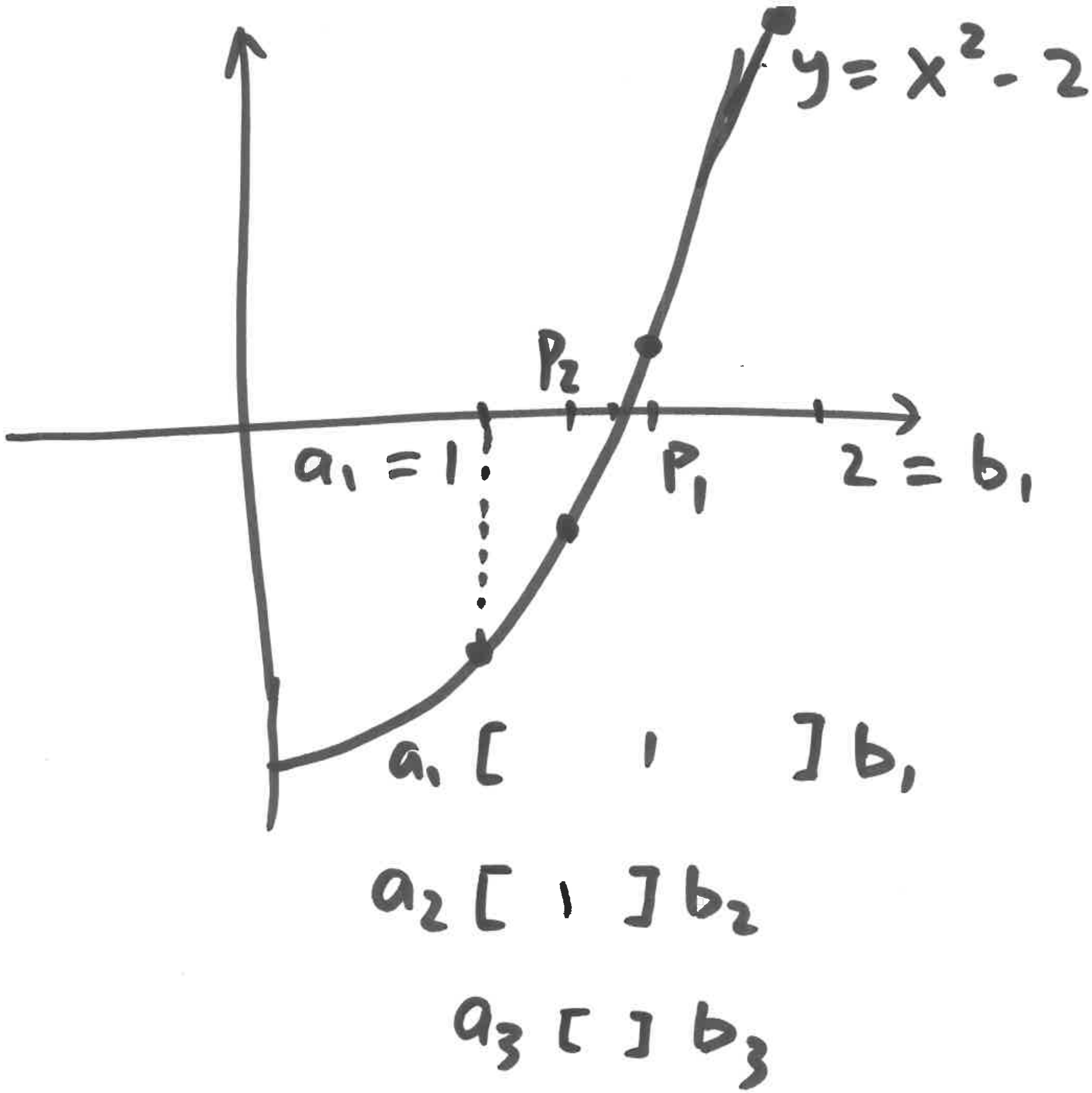
$f(a_3) = -0.4375 \quad f(b_3) = 0.25 \quad f(p_3) = -0.10\cdots$

$[a_4, b_4] = [1.375, 1.5] \quad \boxed{p_4 = 1.4375}$

$\sqrt{2} = 1.414\cdots$

error ≈ 0.023

error bound $= \dfrac{2-1}{2^4} = 0.0625$



$y = x^2 - 2$
p_2
$a_1 = 1$
p_1
$2 = b_1$
a_1 [ ] b_1
a_2 [ ] b_2
a_3 [ ] b_3

__Stability__ (robustness): an algorithm does not break down, even in the presence of round-off errors.

When stability holds for any initial data, then it is called __unconditionally stable__.

Otherwise __conditionally stable__.

- Bisection is unconditionally stable.

Rate of Convergence

Suppose an algorithm generates $\{p_n\}$ to approximate p.

when $\exists\ C > 0$, and a sequence $\{\beta_n\}$, $\beta_n \to 0$, s.t.

$$|p_n - p| \le C \beta_n \quad \forall n \ge 1$$

then the alg. has rate of convergence $O(\beta_n)$.

- For Bisection, $|p_n - p| \le \dfrac{b-a}{2^n}$

the rate of convergence is

$O\left(\dfrac{1}{2^n}\right)$ "linear convergence"