Total time: 50 minutes.

Notice:
(1) Write your solution to each problem on a DIFFERENT answer sheet.
(2) Write your name and your TA's name on every answer sheet.
(3) You can use any method to solve the problems, but you have to justify your answers.
(4) You do not need to simplify your answer (you can leave $62.5 \cdot (10^6 - 7^6)\pi^2$ as final answer, but you cannot leave $\int_0^1 x \, \mathrm{d}x$ as final answer).
(5) You are not allowed to use calculators in this exam.

Problem 1. (20=10+10 points) Let $f(x) = xe^{-x}$.
(1) Determine the interval(s) on which $f(x)$ is decreasing.

$$f'(x) = 1 \cdot e^{-x} + x \cdot (-1)e^{-x} = e^{-x}(1-x)$$

f decreasing, $f'(x) \le 0$, $x \in [1, \infty)$.
(2) Determine the interval(s) on which its graph is concave up.

$$f''(x) = -e^{-x}(1-x) + e^{-x}(-1) = e^{-x}(x-2)$$

f concave up, $f''(x) \ge 0$, $x \in [2, \infty)$.

Problem 2. (30=10+10+10 points) Compute the following definite or indefinite integrals:
(1) $\int_0^3 (x^2-1)^2 \, \mathrm{d}x$

$$= \int_0^3 (x^4 - 2x^2 + 1) \, \mathrm{d}x = (\frac{x^5}{5} - 2\frac{x^3}{3} + x)|_0^3 = \frac{3^5}{5} - 2\frac{3^3}{3} + 3$$

(2) $\int x^2 \sin x \, \mathrm{d}x$

$$= -x^2 \cos x + 2\int x \cos x \, \mathrm{d}x = -x^2 \cos x + 2x \sin x - 2\int \sin x \, \mathrm{d}x = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

(integration by parts twice)

(3) $\int \frac{x^3}{\sqrt{1+x^4}} \, \mathrm{d}x$

$$= \frac{1}{4}\int \frac{1}{\sqrt{u}} \, \mathrm{d}u = \frac{1}{4} 2\sqrt{u} + C = \frac{1}{2}\sqrt{1+x^4} + C$$

(substitution $u = 1 + x^4$)

Problem 3. (25=15+10 points) Let $f(x) = \frac{x}{1+x+x^2}$.

(1) Find the global maximum and global minimum of the function $f(x)$ defined on the interval $[0, 2]$.

$$f'(x) = \frac{(1 + x + x^2) - x(1 + 2x)}{(1 + x + x^2)^2} = \frac{1 - x^2}{(1 + x + x^2)^2}$$

$f'(x) = 0$ gives $x = 1, -1$. Only $x = 1$ is inside $[0, 2]$.

$$f(1) = \frac{1}{3}, \quad f(0) = 0, \quad f(2) = \frac{2}{7}$$

The largest is $f(1) = \frac{1}{3}$ (global max), smallest is $f(0) = 0$ (global min).

(2) Approximate the integral $\int_0^2 f(x)\,\mathrm{d}x$ by using four equally spaced trapezoids.

$\Delta x = (2 - 0)/4 = 0.5$, intermediate points $0, 0.5, 1, 1.5, 2$.

$$\int_0^2 f(x)\,\mathrm{d}x \approx \frac{2 - 0}{2 \cdot 4}(f(0) + 2(f(0.5) + f(1) + f(1.5)) + f(2))$$

$$= \frac{1}{4}(0 + 2(\frac{0.5}{1 + 0.5 + 0.5^2} + \frac{1}{3} + \frac{1.5}{1 + 1.5 + 1.5^2}) + \frac{2}{7})$$

Problem 4. (25=15+10 points) In 2009, Gill and his colleague have provided direct evidence that a shorebird, the Alaskan bar-tailed godwit, makes its eight-day autumn migration from Alaska to New Zealand in one step, with no stopovers to rest or refuel. The minimum requirement for any long flight is that enough fuel is taken on board before departure to sustain the bird for the duration of the flight; in the godwit's case this is about 200 hours. Assuming that the rate of fuel consumption is a fixed proportion of the migration time, the velocity of the migration could be represented by this equation:

$$V = 110 - 0.5t \text{ km/hour}$$

where t is the duration of the flight the bird need to sustain.

(1) The distance the godwit is able to migrate is the antiderivative of V. What is the distance the godwit is able to migrate?

$$\int (110 - 0.5t)\,\mathrm{d}t = 110t - 0.25t^2 + C$$

(2) Find the value of C (appeared in the antiderivative in (1)), if godwits are able to migrate 11,000km in 200 hours.

Substitute distance$= 11000$, $t = 200$,

$$11000 = 110 \cdot 200 - 0.25 \cdot 200^2 + C = 22000 - 10000 + C$$

$C = -1000$.