Review

Ex 1 Let $f(x) = \begin{cases} x^2 & x < 0 \\ x & 0 \le x < 1 \\ \dfrac{2}{x} & x \ge 1 \end{cases}$

Determine points of discontinuity for f.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} x^2 = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x = 0$$

f is cont. @ $x = 0$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \frac{2}{x} = 2$$

$$\neq$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1$$

f is discont. @ $x = 1$

Ex 2 Compute

① $\left(\underline{\sin(e^x)}\cdot \underline{e^{-2x}}\right)'$

$$= \left(\sin\left(\boxed{e^x}\right)\right)'\cdot e^{-2x} + \sin(e^x)\cdot\left(e^{-2x}\right)'$$

$$= \cos(e^x)\cdot e^x \cdot e^{-2x} + \sin(e^x)\cdot e^{-2x}\cdot(-2)$$

$$= \cos(e^x)e^{-x} - 2\sin(e^x)\cdot e^{-2x}$$

② $$\left(\frac{\sin x + e^{2x}}{x^2+1}\right)'$$

$$= \frac{(\cos x + 2e^{2x})\cdot(x^2+1) - (\sin x + e^{2x})\cdot 2x}{(x^2+1)^2}$$

③ $$(e^{\cos x})'' = (\underline{e^{\cos x}}\cdot\underline{(-\sin x)})'$$

$$= e^{\cos x}\cdot(-\sin x)\cdot(-\sin x) + e^{\cos x}\cdot(-\cos x)$$

$$= e^{\cos x}\cdot(\sin^2 x - \cos x).$$

Ex 3 The func. $y(x)$ is defined by

$$x^3 + x + \sin x \cos y = y$$

Compute $\frac{dy}{dx}$

$$3x^2 + 1 + \cos x \cos y + \sin x \cdot (-\sin y) \cdot y' = y'$$

$$3x^2 + 1 + \cos x \cos y = \sin x \cdot \sin y \cdot y' + y'$$

$$\frac{dy}{dx} = y' = \frac{3x^2 + 1 + \cos x \cos y}{\sin x \sin y + 1}$$

__Ex 4__ A circle is increasing its area at a rate of 2. When its area is 10, how fast is the radius increasing?

$A(t)$ $r(t)$

$A = \pi r^2$

Know:

$A'(t_0) = 2$

$A(t_0) = 10$

want:

$r'(t_0) = ?$

$$A' = \pi \cdot 2r \cdot r'$$

$$10 = A(t_0) = \pi \cdot r(t_0)^2$$

$$r(t_0)^2 = \frac{10}{\pi} \qquad r(t_0) = \sqrt{\frac{10}{\pi}}$$

$$r'(t_0) = \frac{A'(t_0)}{\pi \cdot 2 \cdot r(t_0)} = \frac{2}{\pi \cdot 2 \cdot \sqrt{\frac{10}{\pi}}}$$

$$\left(\boxed{r(t)}^2\right)' = 2 \cdot r(t) \cdot r'(t)$$

Ex 5 The amount of material $f(t)$ follows exp. decay. If initially the amount is 5 grams, and after 3 days the amount is 1 gram, how fast is the decay initially?

$$f(t) = \underline{5}\, e^{-kt} \qquad k > 0$$

$\uparrow$

initial amount $f(0)$

$$f(3) = 5e^{-3k} = 1$$

$$e^{-3k} = \frac{1}{5}$$

$$-3k = \ln\frac{1}{5} = -\ln 5$$

$$k = \frac{1}{3}\ln 5$$

$$f(t) = 5e^{-\frac{1}{3}\ln 5 \cdot t}$$

$$f'(t) = 5\cdot\left(-\frac{1}{3}\ln 5\right)e^{-\frac{1}{3}\ln 5 \cdot t}$$

$$f'(0) = -\frac{5}{3}\ln 5$$

decay rate initially is $\frac{5}{3}\ln 5$

Ex 6 Let X be random variable with prob. density function

$$f(x) = \begin{cases} a e^{-x} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

① Determine a.

② $E(X) = ?$

③ $P(X \leq 2) = ?$

① $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{1}^{\infty} a e^{-x}\,dx$$

$$= -a e^{-x} \Big|_{1}^{\infty}$$

$$= -a \Big(\underbrace{\lim_{x\to\infty} e^{-x}}_{=0} - e^{-1} \Big)$$

$$= a e^{-1}$$

$$a e^{-1} = 1 \Rightarrow a = e$$

$$f(x) = \begin{cases} e \cdot e^{-x} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

② $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_{1}^{\infty} x \cdot e \cdot e^{-x}\, dx$

$= e \int_{1}^{\infty} x e^{-x}\, dx$

$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x} dx \end{pmatrix}$

$= e \left[-x e^{-x} \Big|_{1}^{\infty} - \int_{1}^{\infty} (-e^{-x})\, dx \right]$

$$= e\left[-(0 - e^{-1}) - e^{-x}\Big|_1^{\infty}\right]$$

$$= e \cdot \left[e^{-1} - (0 - e^{-1})\right]$$

$$= e\,(e^{-1} + e^{-1}) = 2$$

③ $P(X \leq 2) = \int_{-\infty}^{2} f(x)\,dx$

$= \int_{1}^{2} e \cdot e^{-x}\,dx$

$= -e \cdot e^{-x} \Big|_{1}^{2}$

$= -e(e^{-2} - e^{-1})$

$= -e^{-1} + 1$

Ex 7 Solve $xy' = \frac{x}{y} + \frac{1}{xy}$, $y(1) = -2$

$$x \frac{dy}{dx} = \frac{1}{y}\left(x + \frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{y} \cdot \frac{x + \frac{1}{x}}{x} = \frac{1}{y} \cdot \left(1 + \frac{1}{x^2}\right)$$

$$y\,dy = \left(1 + \frac{1}{x^2}\right) dx$$

$$\int y\,dy = \int \left(1 + \frac{1}{x^2}\right) dx$$

$$\frac{1}{2} y^2 = x - \frac{1}{x} + C$$

$$y^2 = 2\left(x - \frac{1}{x} + C\right)$$

$$y = \pm\sqrt{2\left(x - \frac{1}{x} + C\right)}$$

$$y(1) = -2 \Rightarrow -2 = \pm\sqrt{2(1 - 1 + C)}$$

take "$-$"

$$-2 = -\sqrt{2C}$$

$$4 = 2C \qquad C = 2$$

$$y = -\sqrt{2\left(x - \frac{1}{x} + 2\right)}$$