27.1 ~ 27.3 Population growth, equilibrium

- Logistic growth

$$\frac{dN}{dt} = \underbrace{rN}_{\text{exp. growth}}\underbrace{\left(1 - \frac{N}{K}\right)}_{\text{"restriction"}} \qquad \begin{array}{l} r > 0 \\ K > 0 \end{array}$$

avoid $N(t)$ to grow to more than K

$$\frac{dN}{dt} = \frac{r}{K} N (K - N)$$

$$\frac{dN}{N(K-N)} = \frac{r}{K} dt$$

$$\frac{1}{N(K-N)} = \frac{1}{K} \left(\frac{1}{N} + \frac{1}{K-N} \right)$$

$$\frac{1}{\cancel{K}} \left(\frac{1}{N} + \frac{1}{K-N} \right) dN = \frac{r}{\cancel{K}} dt$$

$$\left(\frac{1}{N} + \frac{1}{K-N} \right) dN = r\,dt$$

$$\int \left(\frac{1}{N} + \frac{1}{K-N} \right) dN = \int r \, dt$$

$$\ln N - \ln(K-N) = rt + C$$

needs $N > 0$ needs $K-N > 0$

$$\int \frac{1}{K-N} dN = -\int \frac{1}{u} du = -\ln u + C$$

$$= -\ln(K-N) + C$$

$u = K-N \quad du = -dN$

$$\ln \frac{N}{K-N} = rt + C$$

$$\frac{N}{K-N} = e^{rt+C}$$

$$N = Ke^{rt+C} - Ne^{rt+C}$$

$$(1+e^{rt+C})N = Ke^{rt+C}$$

$$N = \frac{Ke^{rt+C}}{1+e^{rt+C}} = \frac{K}{e^{-rt-C}+1}$$

Given $N(0) = N_0$

$$N_0 = \frac{K}{e^{-r \cdot 0 - c} + 1}$$

$$\Rightarrow \cdots \Rightarrow N(t) = \frac{K N_0}{N_0 + (K - N_0) e^{-rt}}$$

$$\lim_{t \to \infty} N(t) = K$$

- Gompertz growth (for tumor).

$$\underbrace{\frac{1}{V}\frac{dV}{dt}}_{\text{specific growth rate}} = k e^{-\alpha t} \qquad \begin{array}{l} k > 0 \\ \alpha > 0 \end{array}$$

$$\frac{dV}{V} = k e^{-\alpha t} dt$$

$$\int \frac{dV}{V} = \int k e^{-\alpha t} dt$$

$$\ln V = k \frac{1}{-\alpha} e^{-\alpha t} + C$$

$$V = e^{-\frac{k}{\alpha}e^{-\alpha t} + C}$$

$$= \exp\left(-\frac{k}{\alpha}e^{-\alpha t} + C\right)$$

Given $V(0) = V_0$

$$V_0 = \exp\left(-\frac{k}{\alpha}\cancel{e^{-\alpha \cdot 0}} + C\right)$$

$$\ln V_0 = -\frac{k}{\alpha} + C$$

$$C = \ln V_0 + \frac{k}{\alpha}$$

$$V = \exp\left(-\frac{k}{\alpha} e^{-\alpha t} + \ln V_0 + \frac{k}{\alpha}\right)$$

$$= \exp\left(\frac{k}{\alpha}\left(1 - e^{-\alpha t}\right)\right) \cdot \exp\left(\ln V_0\right)$$

$$= V_0 \exp\left(\frac{k}{\alpha}\left(1 - e^{-\alpha t}\right)\right)$$

$$\lim_{t \to \infty} V(t) = V_0 \exp\left(\frac{k}{\alpha}\right)$$

- <u>Equilibrium</u>

RHS indep. of t

For an autonomous differential equation

$$\frac{dN}{dt} = F(N)$$

the possible values of $\lim_{t \to \infty} N(t)$ are given by $F(N) = 0$

"equilibrium"

- **Stability of equilibrium.**

$$\frac{dN}{dt} = F(N)$$

For an equilibrium $N = N_1$,

if $\lim_{t \to \infty} N(t) = N_1$ whenever

N_0 is close to N_1, then

N_1 is <u>stable</u>, otherwise

<u>unstable</u>

<u>Ex</u> $$\frac{dN}{dt} = F(N) = rN\left(1 - \frac{N}{K}\right)$$

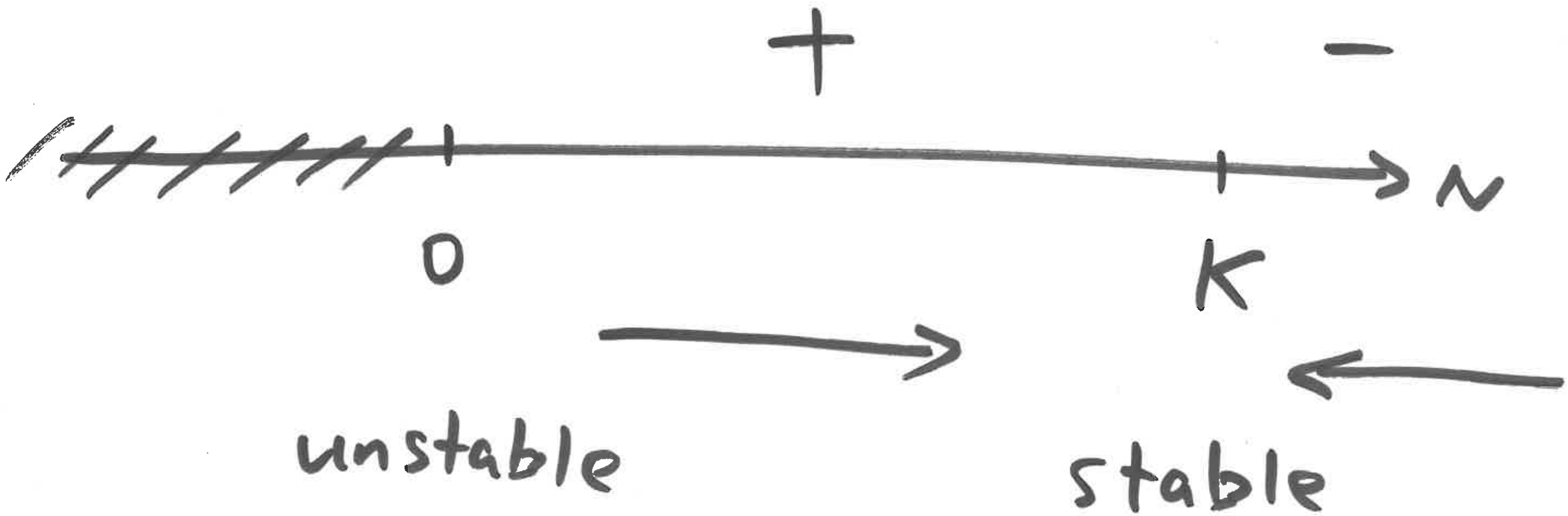
$$F(N) = 0 \Rightarrow \underline{N = 0} \text{ or } 1 - \frac{N}{K} = 0$$

no popu. at all

$$\underline{N = K}$$

stay at "enviromental constraint"

$$\frac{dN}{dt} = F(N)$$



Ex Determine equilibria and their stability

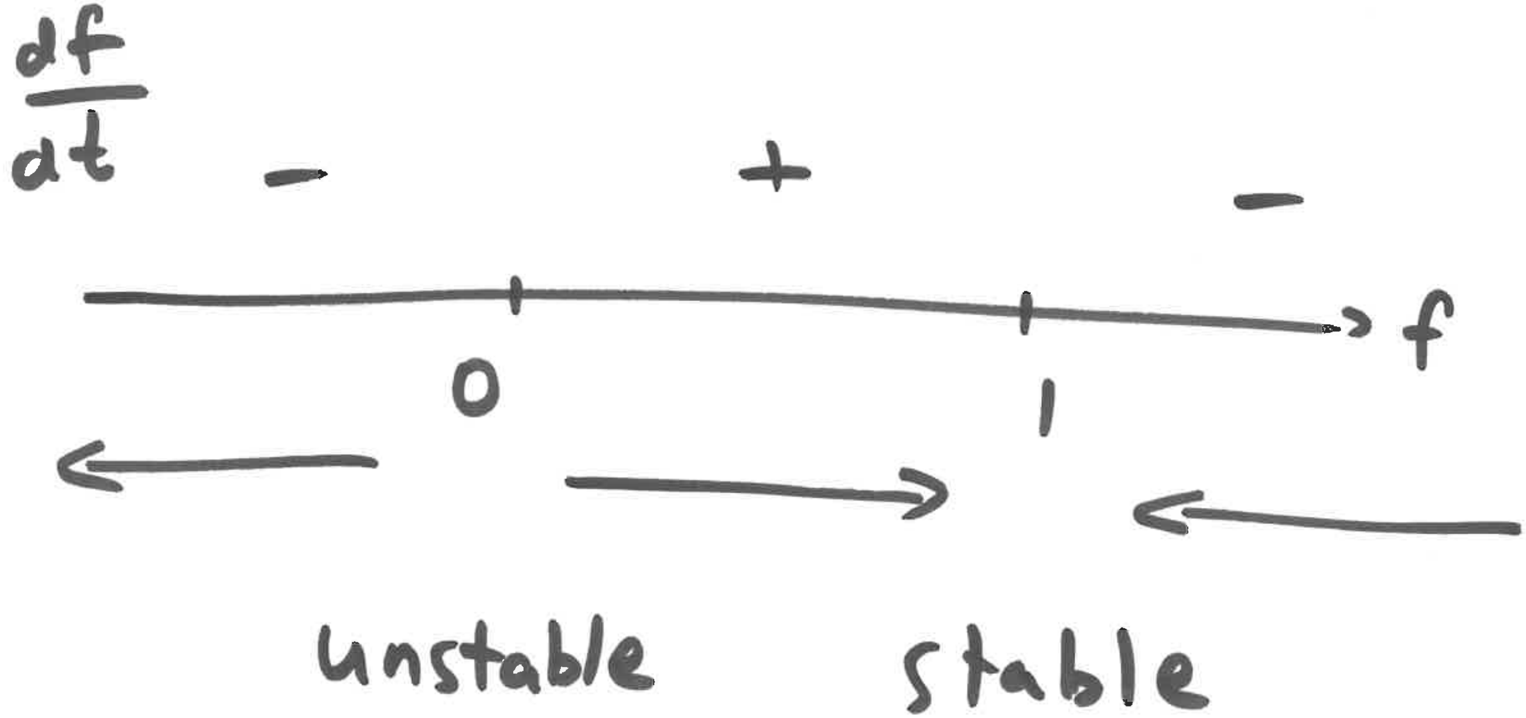
(1) $$\frac{df}{dt} + f^2 = f$$

$$\frac{df}{dt} = -f^2 + f$$

$$-f^2 + f = 0 \implies f^2 - f = 0$$

$$f(f-1) = 0$$

$$f = 0 \text{ or } f = 1.$$

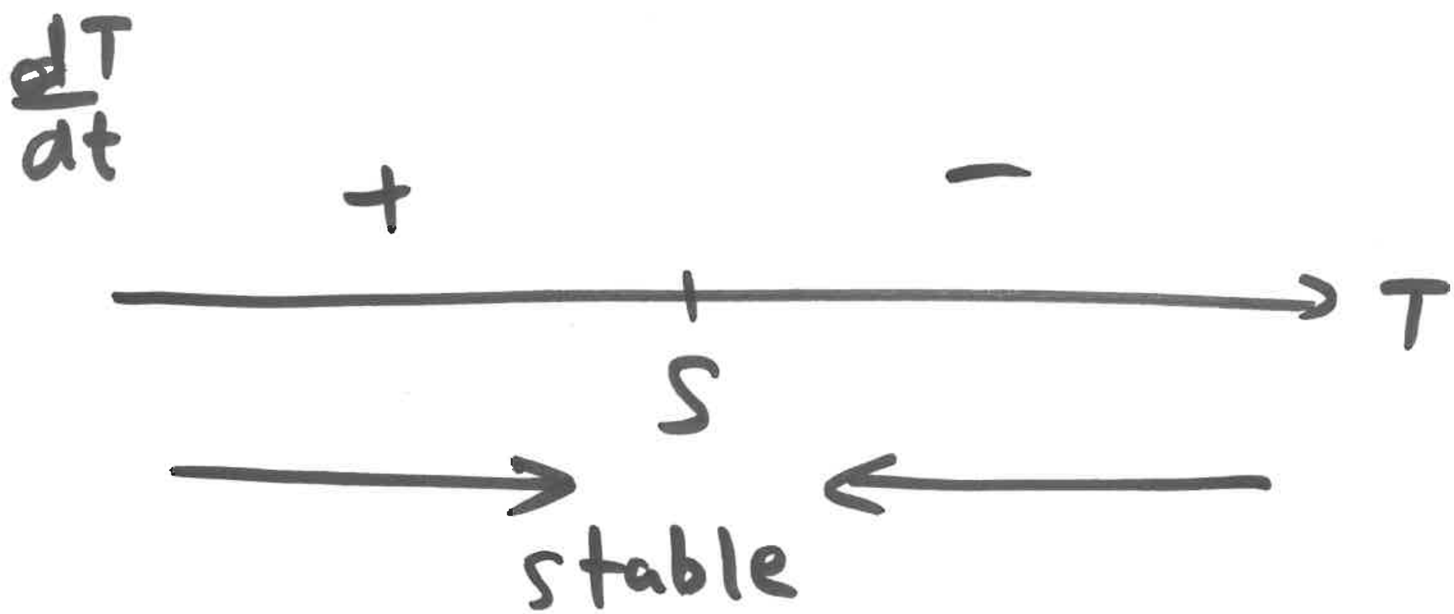

df/dt
-
+
-
f
0
1
unstable
stable

② $$\frac{dT}{dt} = k(S - T)$$

$T(t)$: body temperature

$k > 0$, $S > 0$

$k(S - T) = 0$ $\quad$ $T = S$



$$\frac{dT}{S-T} = k\,dt$$

$$\int \frac{dT}{S-T} = \int k\,dt$$

$$-\ln(S-T) = kt + C$$

$T < S$

$$\ln(S-T) = -kt - C$$

$$S - T = e^{-kt-C} = C_1 e^{-kt}$$

$$T = S - C_1 e^{-kt}$$

Given $T(0) = T_0$

$$T_0 = S - c_1 \cancel{e^{-k \cdot 0}}$$

$$c_1 = S - T_0$$

$$T = S - (S - T_0)e^{-kt}$$

$$\lim_{t \to \infty} T(t) = S$$