26.1 Separation of variables

- Differential equations for $y(t)$

$$\frac{dy}{dt} = \boxed{\text{something w/ } y, t}$$

ex. $$\frac{dy}{dt} = y + t$$

$$\frac{dy}{dt} = y^2 \qquad \frac{dy}{dt} = \frac{\sin y + 2t}{t^2 \cdot y}$$

- Separation of variables

$$\frac{dy}{dt} = h(y)\,g(t)$$

$$\frac{dy}{h(y)} = g(t)\,dt$$

$$\int \frac{dy}{h(y)} = \int g(t)\,dt$$

$$H(y) = G(t) + C$$

solve for y gives $y =$ func. of t.

Ex 1 Solve

① $$\frac{dy}{dt} = t^2 \cdot y^2$$

$$\frac{dy}{y^2} = t^2\,dt$$

$$\int \frac{dy}{y^2} = \int t^2\,dt$$

$$-\frac{1}{y} = \frac{1}{3}t^3 + C$$

$$\frac{1}{y} = -\left(\frac{1}{3}t^3 + C\right)$$

$$\boxed{y = -\frac{1}{\frac{1}{3}t^3 + C}}$$

(2) $$\frac{dy}{dt} = \frac{\sin t}{y}$$

$$y\,dy = \sin t\,dt$$

$$\int y\,dy = \int \sin t\,dt$$

$$\frac{1}{2}y^2 = -\cos t + C$$

$$y^2 = 2(C - \cos t)$$

$$y = \pm\sqrt{2(C - \cos t)}$$

③ $$\frac{dy}{dt} = ty$$

in fact

$$\int \frac{dy}{y} = \ln|y| + C$$

$$\frac{dy}{y} = t\,dt$$

$$\int \frac{dy}{y} = \int t\,dt$$

$$\ln y = \frac{1}{2}t^2 + C$$

$$y = e^{\frac{1}{2}t^2 + C} = e^C \cdot e^{\frac{1}{2}t^2} = C_1 e^{\frac{1}{2}t^2}$$

allow $C_1 < 0$ to handle $y(t) < 0$

④ $$\frac{dy}{dt} = \frac{1}{y+1} \cdot 1$$

$$(y+1)\,dy = 1 \cdot dt$$

$$\int (y+1)\,dy = \int 1 \cdot dt$$

$$\frac{1}{2}y^2 + y = t + C$$

$$\frac{1}{2}y^2 + y - (t+C) = 0$$

$$y = \frac{-1 \pm \sqrt{1^2 - 4 \cdot \frac{1}{2} \cdot (-(t+C))}}{2 \cdot \frac{1}{2}}$$

$$= -1 \pm \sqrt{1 + 2(t + C)}$$

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5 $t \dfrac{dy}{dt} = y^3 \qquad (t > 0)$

$$\frac{dy}{dt} = y^3 \cdot \frac{1}{t}$$

$$\frac{dy}{y^3} = \frac{1}{t} dt$$

$$\int \frac{dy}{y^3} = \int \frac{1}{t} dt$$

$$\frac{1}{-2} \cdot \frac{1}{y^2} = \ln t + C$$

$$\frac{1}{y^2} = -2(\ln t + C)$$

$$y^2 = -\frac{1}{2(\ln t + C)}$$

$$y = \pm\sqrt{-\frac{1}{2(\ln t + C)}}$$

- Diff. eq. w/ initial/boundary conditions $y(t_0) = y_0$

Substitute into the final expression, solve for C

<u>Ex 2</u> Solve

① (exp. growth)

$$\frac{dy}{dt} = 2y \cdot 1 \quad \text{w/} \quad y(0) = 10$$

$$\frac{dy}{2y} = dt$$

$$\int \frac{dy}{2y} = \int dt$$

$$\frac{1}{2} \ln y = t + C$$

$$\ln y = 2(t+C)$$

$$y = e^{2t+2C} = e^{2C} \cdot e^{2t} = C_1 e^{2t}$$

$$y(0)=10 \Rightarrow 10 = C_1 e^{2\cdot 0} = C_1$$

$$\boxed{y = 10e^{2t}}$$

② $\frac{dy}{dt} = t + yt$ w/ $y(1) = 2$

$$\frac{dy}{dt} = t \cdot (1+y)$$

$$\frac{dy}{1+y} = t\,dt$$

$$\int \frac{dy}{1+y} = \int t\,dt$$

$$\ln(1+y) = \frac{1}{2}t^2 + C$$

$$1 + y = e^{\frac{1}{2}t^2 + C}$$

$$y = e^{\frac{1}{2}t^2 + C} - 1$$

$$y(1) = 2 \Rightarrow 2 = e^{\frac{1}{2}1^2 + C} - 1$$

$$e^{\frac{1}{2} + C} = 3$$

$$\frac{1}{2} + C = \ln 3 \qquad C = \ln 3 - \frac{1}{2}$$

$$\boxed{y = e^{\frac{1}{2}t^2 + \ln 3 - \frac{1}{2}} - 1}$$

③ $xy' - (2x+1)e^{-y} = 0$

w/ $y(1) = 2$

$$x\frac{dy}{dx} - (2x+1)e^{-y} = 0$$

$$x\frac{dy}{dx} = (2x+1)e^{-y}$$

$$\frac{dy}{dx} = \frac{1}{x}\cdot(2x+1)\cdot e^{-y}$$

$$\frac{dy}{e^{-y}} = \frac{1}{x}(2x+1)\,dx$$

$$\frac{1}{e^{-y}} = e^{y} \qquad \frac{1}{x}(2x+1) = 2 + \frac{1}{x}$$

$$\int e^{y}\,dy = \int \left(2 + \frac{1}{x}\right)dx$$

$$e^{y} = 2x + \ln x + C$$

$$y = \ln(2x + \ln x + C)$$

$$y(1) = 2 \implies 2 = \ln(2\cdot 1 + \ln 1 + C)$$
$$= \ln(2 + C)$$
$$e^2 = 2 + C$$
$$C = e^2 - 2$$
$$\boxed{y = \ln(2x + \ln x + e^2 - 2)}$$

(4) $\frac{dy}{dt} = y \ln y$ w/ $y(0) = 2$

$$\frac{dy}{y \ln y} = dt$$

$$\int \frac{dy}{y \ln y} = \int dt$$

$$\int \frac{dy}{y \ln y} = \int \frac{1}{u}\, du = \ln u + C$$

$$= \ln(\ln y) + C$$

$u = \ln y \quad du = \frac{1}{y}\, dy$

$$\ln(\ln y) = t + C$$

$$\ln y = e^{t+C} = e^{C} \cdot e^{t} = C_1 e^{t}$$

$$y = e^{C_1 e^{t}}$$

$$y(0) = 2 \Rightarrow 2 = e^{C_1 e^{0}} = e^{C_1}$$

$$\ln 2 = C_1$$

$$y =$$

$$e^{(\ln 2) \cdot e^{t}}$$

(5) $\frac{dy}{dt} = y^2$ w/ $y(0) = 2$

$$\frac{dy}{y^2} = dt$$

$$\int \frac{dy}{y^2} = \int dt$$

$$-\frac{1}{y} = t + C$$

$$\frac{1}{y} = -(t+C) \qquad y = -\frac{1}{t+C}$$

$$y(0) = 2 \implies 2 = -\frac{1}{0 + C}$$

$$2 = -\frac{1}{C}$$

$$\frac{1}{C} = -2 \qquad C = -\frac{1}{2}$$

$$y = -\frac{1}{t - \frac{1}{2}} = \frac{1}{\frac{1}{2} - t}$$