23.2 Integration by parts

$$(\underline{x}\,\underline{e^{2x}})' = 1 \cdot e^{2x} + x \cdot 2e^{2x}$$

$$x e^{2x} = \int e^{2x}\,dx + \int x \cdot 2e^{2x}\,dx$$

$$\int \underbrace{x \cdot 2e^{2x}\,dx}_{\text{hard}} = x e^{2x} - \underbrace{\int e^{2x}\,dx}_{\text{easy}}$$

$$\left(f(x)\, g(x) \right)' = f'(x)\, g(x) + f(x)\, g'(x)$$

$$f(x)\, g(x) = \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

$$\boxed{\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx}$$

- Write $u = f(x)$, $v = g(x)$

$$du = f'(x)\, dx, \quad dv = g'(x)\, dx$$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Ex 1

① $\int \underline{x}\, \underline{e^{2x}\, dx}$

$$\left(\begin{array}{ll} u = x & v = \frac{1}{2}e^{2x} \\ du = 1 \cdot dx & dv = e^{2x} dx \end{array} \right)$$

$$= x \cdot \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x} \cdot 1 \cdot dx$$

$$= \frac{1}{2}x e^{2x} - \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C$$

② $\int \underline{x}\, \underline{\sin x \, dx}$

$$\left(\begin{array}{ll} u = x & v = -\cos x \\ du = dx & dv = \sin x \, dx \end{array} \right)$$

$$= x \cdot (-\cos x) - \int (-\cos x) \, dx$$

$$= -x \cos x + \sin x + C$$

③ $\int \underbrace{\ln x}\, \underbrace{dx}$

$$\left(\begin{array}{ll} u = \ln x & v = x \\ du = \frac{1}{x}\,dx & dv = 1 \cdot dx \end{array} \right)$$

$$= \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$= x \ln x - x + C$$

- Choice of u and dv

u	$\ln x$
	polynomials
v	e^x, $\sin x$, $\cos x$

Ex 2

① $$\int e^{-x} x^2 \, dx = \int x^2 e^{-x} \, dx$$

$$\left(\begin{array}{ll} u = x^2 & v = -e^{-x} \\ du = 2x\,dx & dv = e^{-x}\,dx \end{array} \right)$$

$$= x^2 \cdot (-e^{-x}) - \int (-e^{-x}) \cdot 2x \, dx$$

$$= -x^2 e^{-x} + 2 \int x e^{-x} \, dx$$

$$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x}\,dx \end{pmatrix}$$

$$= -x^2e^{-x} + 2\left(x\cdot(-e^{-x}) - \int(-e^{-x})\,dx \right)$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

- $\int e^{-x^3} x^2 \, dx = \dots\dots$

  $u = -x^3$ (subs.)

- $\int e^{-x} x^2 \, dx$ Never do this

$$\left(\begin{array}{ll} u = e^{-x} & v = \frac{1}{3}x^3 \\ du = -e^{-x} dx & dv = x^2 dx \end{array} \right)$$

$$= e^{-x} \cdot \frac{1}{3}x^3 - \int \underbrace{\frac{1}{3}x^3}_{\text{harder than before!}} \cdot (-e^{-x}) \, dx$$

(2) $\int \underline{(2x+3)}\,\underline{\cos x\,dx}$

$$\left(\begin{array}{ll} u = 2x+3 & v = \sin x \\ du = 2\,dx & dv = \cos x\,dx \end{array}\right)$$

$$= (2x+3)\sin x - \int \sin x \cdot 2\,dx$$

$$= (2x+3)\sin x + 2\cos x + C$$

③ $\int \frac{\ln x}{x^3}\,dx = \int \underline{\ln x} \cdot \underline{\frac{1}{x^3}\,dx}$

$$\left(\begin{array}{ll} u = \ln x & v = \frac{1}{-2}\cdot\frac{1}{x^2} \\ du = \frac{1}{x}\,dx & dv = \frac{1}{x^3}\,dx \end{array} \right)$$

$$= \ln x \cdot \left(-\frac{1}{2}\cdot\frac{1}{x^2}\right) - \int \left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\cdot\frac{1}{x^2}\right) + C$$

④ $\int x^3 e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\cdot x^2 \cdot \underline{2x\,dx}$

$\left(y = x^2 \qquad dy = 2x\,dx \right)$

$= \frac{1}{2}\int e^y \cdot y\,dy = \frac{1}{2}\int \underline{y}\ \underline{e^y dy}$

$\left(\begin{array}{ll} u = y & v = e^y \\ du = dy & dv = e^y dy \end{array} \right)$

$= \frac{1}{2}\left(y e^y - \int e^y\,dy \right)$

$$= \frac{1}{2} y e^{y} - \frac{1}{2} e^{y} + C$$

$$= \frac{1}{2} x^{2} e^{x^{2}} - \frac{1}{2} e^{x^{2}} + C$$

(5) $\int \sin(\sqrt{x})\,dx = \int \sin(\sqrt{x})\cdot 2\sqrt{x}\,\underline{\frac{1}{2\sqrt{x}}\,dx}$

$\left(y = \sqrt{x} \qquad dy = \frac{1}{2\sqrt{x}}\,dx \right)$

$= \int \sin(y)\cdot 2y\,dy = 2\int \underline{y}\ \underline{\sin y\,dy}$

$\begin{pmatrix} u = y & v = -\cos y \\ du = dy & dv = \sin y\,dy \end{pmatrix}$

$= 2\left(y\cdot(-\cos y) - \int(-\cos y)\,dy \right)$

$$= -2y\cos y + 2\sin y + C$$

$$= -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C$$

- Integration by parts for definite integrals

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx$$

<u>Ex 3</u> $\int_0^1 \underline{x}\,\underline{e^{-x}\,dx}$

$$\begin{pmatrix} u = x & v = -e^{-x} \\ du = dx & dv = e^{-x}\,dx \end{pmatrix}$$

$$= x(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})\,dx$$

$$= 1 \cdot (-e^{-1}) - 0 + (-e^{-x})\Big|_0^1$$

$$= -e^{-1} - e^{-1} - (-e^{-0})$$

$$= 1 - 2e^{-1}$$