23.1 Substitution method

$$(e^{ax})' = e^{ax} \cdot a \qquad \int e^{ax} \cdot a \, dx = e^{ax} + C$$

$$(\sin(ax+b))' = \cos(ax+b) \cdot a \qquad \int \cos(ax+b) \cdot a \, dx = \sin(ax+b) + C$$

$$(e^{x^2})' = e^{x^2} \cdot 2x \qquad \int e^{x^2} \cdot 2x \, dx = e^{x^2} + C$$

- Substitution method

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C.$$

- Write $u = g(x)$, $du = g'(x)\, dx$

$$\left(\frac{du}{dx} = g'(x)\right)$$

$$\int f'(u)\, du$$

Ex 1

① $\int e^{\boxed{x^2}} \cdot \underline{2x\,dx}$ want to write as $\underline{f'(u)}\,\underline{du}$

$\left(u = x^2 \qquad du = 2x\,dx\right)$

$= \int e^u\,du = e^u + C$

$= e^{x^2} + C$

② $\int \cos(\boxed{ax+b}) \cdot a\,dx$

$(u = ax+b \qquad du = a\,dx)$

$= \int \cos u \, du = \sin u + C$

$= \sin(ax+b) + C$

3 $\displaystyle\int \frac{1}{\boxed{x^3+1}}\, \underline{3x^2\,dx}$

$$\left(u = x^3+1 \qquad du = 3x^2\,dx\right)$$

$$= \int \frac{1}{u}\,du = \ln u + C$$

$$= \ln(x^3+1) + C$$

- Sometimes you need to multiply constant to make "du"

Ex 2

① $\int e^{x^2}\, x\,dx$

$(u = x^2 \quad du = 2x\,dx)$

$= \frac{1}{2}\int e^{\boxed{x^2}}\, \underline{2x\,dx} = \frac{1}{2}\int e^{u}\,du$

$= \frac{1}{2}e^{u} + C = \frac{1}{2}e^{x^2} + C$

② $\int \frac{1}{(2x-3)^3} dx$

$\left(u = 2x-3 \qquad du = 2\,dx \right)$

$= \frac{1}{2} \int \frac{1}{(2x-3)^3} \, \underline{2\,dx}$

$= \frac{1}{2} \int \frac{1}{u^3} du = \frac{1}{2} \cdot \frac{1}{-2} \frac{1}{u^2} + C$

$= -\frac{1}{4} \cdot \frac{1}{(2x-3)^2} + C$

- u is usually a building block of a complicated part,

du is usually a simple factor (appears on numerator for fractions)

Ex 3

$\sqrt{1+x^3}$ is complicated

① $\int \underset{\sim}{x^2} \sqrt{\boxed{1+x^3}}\, \underline{dx}$

$\left(u = 1+x^3 \qquad du = 3x^2 dx\right)$

$= \frac{1}{3} \int \sqrt{1+x^3}\, \underline{3x^2 dx}$

$= \frac{1}{3} \int \sqrt{u}\, du = \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$

$= \frac{2}{9} (1+x^3)^{\frac{3}{2}} + C$

② $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}} \cdot x\,dx$

$$\left(u = 1-x^2 \qquad du = -2x\,dx\right)$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \cdot \underline{(-2)x\,dx}$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$$

$$= -\sqrt{1-x^2} + C$$

③ $\int (x^2+2x-1)^{10}(x+1)\,dx$

$\left(u = x^2+2x-1 \qquad du = (2x+2)\,dx\right)$

$= \frac{1}{2}\int (x^2+2x-1)^{10}\underline{(2x+2)dx}$

$= \frac{1}{2}\int u^{10}\,du = \frac{1}{2}\cdot\frac{1}{11}u^{11} + C$

$= \frac{1}{22}(x^2+2x-1)^{11} + C$

④ $\int \sin^3 x \cos x \, dx$

$(u = \sin x \qquad du = \cos x \, dx)$

$= \int u^3 \, du = \frac{1}{4} u^4 + C$

$= \frac{1}{4} \sin^4 x + C$

5) $\int e^{\cos x} \sin x \, dx$

$\left(u = \cos x \qquad du = -\sin x \, dx\right)$

$= -\int e^{\cos x} \underline{(-\sin x)\, dx}$

$= -\int e^{u} \, du = -e^{u} + C$

$= -e^{\cos x} + C$

(6) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$= \int \frac{1}{\cos x} \sin x \, dx$

$(u = \cos x \qquad du = -\sin x \, dx)$

$= -\int \frac{1}{\cos x} \underline{(-\sin x) \, dx}$

$= -\int \frac{1}{u} \, du = -\ln u + C$

$= -\ln(\cos x) + C$

- For $\ln x$, you may need

$u = \ln x$, $du = \frac{1}{x}\,dx$

<u>Ex 4</u> $\int \frac{\ln x}{x}\,dx = \int \ln x \cdot \frac{1}{x}\,dx$

$(u = \ln x \qquad du = \frac{1}{x}\,dx)$

$= \int u\,du = \frac{1}{2}u^2 + C$

$= \frac{1}{2}(\ln x)^2 + C$

- when every x is on exponential, try $u = e^x$ or similar

Ex 5

① $\int \frac{e^x}{e^x+1}\,dx = \int \frac{1}{e^x+1}\,e^x\,dx$

$(u = e^x+1 \quad du = e^x\,dx)$

$= \int \frac{1}{u}\,du = \ln u + C = \ln(e^x+1)+C$

② $\int \sqrt{e^{3x} - e^{2x}}\, dx$

$\left(u = e^x - 1 \quad du = e^x dx \right)$

$= \int \sqrt{e^{2x}(e^x - 1)}\, dx$

$= \int \sqrt{e^x - 1}\; \underline{e^x\, dx}$

$= \int \sqrt{u}\, du = \frac{2}{3} u^{\frac{3}{2}} + C$

$= \frac{2}{3} (e^x - 1)^{\frac{3}{2}} + C$

- Substitution for definite integrals.

$$\int_a^b f'(g(x))\,g'(x)\,dx$$

$$= \int_{g(a)}^{g(b)} f'(u)\,du$$

$u = g(x)$

$du = g'(x)\,dx$

$x : [a, b]$

$u = g(x) : [g(a), g(b)]$

Ex 6 $\int_0^{\pi} x \sin(x^2)\, dx$

$$\left(u = x^2 \qquad du = 2x\, dx \right)$$

$$= \frac{1}{2}\int_0^{\pi} \sin(x^2) \cdot \underline{2x\, dx}$$

$$\boxed{\begin{array}{r} x : [0, \pi] \\ u = x^2 : [0, \pi^2] \end{array}}$$

$$= \frac{1}{2} \int_0^{\pi^2} \sin u \, du$$

$$= \frac{1}{2} (-\cos u) \Big|_0^{\pi^2} = \frac{1}{2}\left(-\cos \pi^2 - (-\cos 0) \right)$$

$$= \frac{1}{2} \left(1 - \cos \pi^2 \right)$$